The diffeomorphism group of a non-compact orbifold

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We endow the diffeomorphism group $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ of a paracompact (reduced) orbifold with the structure of an infinite dimensional Lie group modelled on the space of compactly supported sections of the tangent orbibundle. For a second countable orbifold, we prove that $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ is C^0-regular and thus regular in the sense of Milnor. Furthermore an explicit characterization of the Lie algebra of $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ is given.

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1. Introduction and statement of results

Diffeomorphism groups of compact manifolds and their subgroups are prime examples of infinite dimensional Lie groups. There are many well known results concerning the Lie group structure of these groups, e.g.: A classical result states that each diffeomorphism group of a compact manifold is an infinite dimensional regular Lie group (see [46]). For the algebraic structure of these groups see [4]. More generally, Lie group structures on diffeomorphism groups of paracompact manifolds (even with corners) were constructed in [45] (also cf. [26] for the special case $\mathrm{Diff}(\mathbb{R}^n)$). Furthermore in [42] the diffeomorphism groups of manifolds were endowed with the structure of a regular Lie group in the "covenient setting of analysis". We remark that the "convenient setting of analysis" (see [42]) is inequivalent to the setting of analysis adopted in this paper. Our studies are based on a concept of C^r-maps between locally convex spaces known as Keller's C^r_c-theory [38] (see [46], [22] and [30] for streamlined expositions, cf. also [5]). The present paper generalizes the results on diffeomorphism groups of manifolds to diffeomorphism groups of reduced paracompact orbifolds.

Orbifolds were first introduced by Satake in [52] as *V-manifolds* to generalize the concept of a manifold. Later on they appear in the works of Thurston (cf. [53]), who popularized the term "orbifold". One might think of an orbifold as a manifold with "mild singularities". Objects with orbifold structure arise naturally, for example in symplectic geometry, physics and algebraic geometry (cf. the survey in [1]). It is well known that there are at least three different ways to define an orbifold: Orbifolds may be described by atlases of local charts akin to a manifold (see [1, 32, 48]). Furthermore orbifolds correspond to special classes of Lie groupoids (see [48] or the survey [47]). Finally one might think of them as Deligne-Mumford stacks (cf. [44]). The author thinks that the first approach is suited best to apply methods from differential geometry to orbifolds. Hence in the present paper we define orbifolds in local charts. Unfortunately this point of view makes it difficult to define morphisms of orbifolds. The literature proposes a variety of notions for these morphisms, e.g. the Chen-Ruan good map [14], the Moerdijk-Pronk strong map [49], or the maps in [6]. However, orbifolds in local charts are equivalent to certain Lie groupoids, whose morphisms are well understood objects. Thus orbifold morphisms should correspond to a class of Lie groupoid morphisms. The orbifold maps introduced by Pohl in [51] satisfy these requirements, since they were modelled to be equivalent to groupoid morphisms.[1] Furthermore these maps allow a characterization in local charts, which is amenable to methods of differential geometry and Lie theory. Therefore in the present paper, maps of orbifolds will be orbifold maps in the sense of Pohl [51] (see Appendix E for a comprehensive introduction to these maps).
To construct the Lie group structure on the diffeomorphism group of an orbifold we have to develop several tools from Riemannian geometry on orbifolds. These results are of interest in their own right and include the following:
We discuss geodesics on Riemannian orbifolds and prove that they are uniquely determined by their starting values. Then a detailed construction for a *Riemannian orbifold exponential map* $[\exp_{\mathrm{Orb}}]$ is provided. This map is an orbifold morphism in the sense of Pohl [51], which generalises the concept of a Riemannian exponential map to Riemannian orbifolds (cf. [32] respectively [14] for Riemannian exponential maps on geodesically complete orbifolds).

[1] Other concepts of orbifold maps also satisfy similar properties, cf. [1, Section 2.4]

The Riemannian exponential map on a manifold may be used to construct the Lie group structure on the diffeomorphism group of that manifold (cf. [46]). The Riemannian orbifold exponential map allows us to follow this line of thought: We endow the diffeomorphism group of a paracompact reduced orbifold with the structure of an infinite dimensional locally convex Lie group in the sense of [50]. More precisely the main results subsume the following theorem (cf. Theorem 6.2.4):

Theorem A *The diffeomorphism group* $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ *of a paracompact reduced orbifold* $(Q,\mathcal{U})$ *can be made into a Lie group in a unique way, such that the following is satisfied:*
For some Riemannian orbifold metric ρ *on* $(Q,\mathcal{U})$, *let* $[\exp_{\mathrm{Orb}}]$ *be the Riemannian orbifold exponential map. There exists an open zero-neighborhood* $\mathcal{H}_\rho$ *in the space of compactly supported sections of the tangent orbibundle, such that*

$$E\colon \mathcal{H}_\rho \to \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U}), [\hat{\sigma}] \mapsto [\exp_{\mathrm{Orb}}] \circ [\hat{\sigma}]$$

induces a well defined C^∞*-diffeomorphism onto an open submanifold of* $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$. *The condition is then satisfied for every Riemannian orbifold metric on* $(Q,\mathcal{U})$. *If* $(Q,\mathcal{U})$ *is a compact orbifold, then the Lie group* $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ *is a Fréchet Lie group.*

This result generalizes the classical construction of a Lie group structure on the diffeomorphism group $\mathrm{Diff}(M)$ of a paracompact manifold. For such a manifold we may consider subgroups of $\mathrm{Diff}(M)$, whose elements coincide outside of a given compact set with the identity. It is known that these subgroups are Lie subgroups of $\mathrm{Diff}(M)$ (cf. [25, Section 14]). Section 6.2 contains a similar result for diffeomorphisms of orbifolds, which is a consequence of Theorem A:

Proposition B *Let* $(Q,\mathcal{U})$ *be a paracompact reduced orbifold. For each compact subset* K *of* Q *we define the group* $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_K$ *of all orbifold diffeomorphisms which coincide off* K *with the identity morphism of the orbifold. Let* $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_c$ *be the group of all orbifold diffeomorphisms which coincide off some compact set with the identity morphism of the orbifold. Then the following holds:*

(a) *The group* $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_c$ *is an open Lie subgroup of* $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$.
(b) *For each compact subset* K *of* Q, *there is a compact set* $L \supseteq K$ *such that* $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_L$ *is a closed Lie subgroup of* $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$. *The closed Lie subgroup* $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_L$ *is modelled on the space of sections in the tangent orbibundle, which vanish off* L.

If $(Q,\mathcal{U})$ *is a trivial orbifold (i.e. a manifold), one may always choose* $K = L$ *in* (b).

We remark that Lie group structures for diffeomorphism groups of orbifolds were already considered by Borzellino and Brunsden. In [6] and the follow up [7], the diffeomorphism group of a compact orbifold has been turned into a convenient Fréchet Lie group. The author does not know whether the orbifold morphisms introduced in [6] are equivalent to the class orbifold maps considered in the present paper. If both notions were equivalent, the results of [6, 7] concerning the Lie group structure of the diffeomorphism group are subsumed in Theorem A. This follows from the fact that

in the Fréchet setting both notions of "smooth maps" coincide (cf. [38] and [42, Theorem 4.11 (a)]). Hence Fréchet Lie groups in the sense of [50] and "convenient Fréchet Lie groups" coincide. However, we have to point out that the exposition in [6] contains several major errors (see Remark 6.2.9 for further information on this topic).
We also mention that in the groupoid setting, topologies for spaces of orbifold maps have been considered. Chen constructs in [13] a topology on the space of orbifold morphisms whose domain is a compact orbifold, turning the space into a Banach orbifold (also cf. similar results in [33]). The exposition of the present paper is independent of these results.

After constructing the Lie group $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$, we characterize the Lie algebra associated to this group. It is instructive to recall the special case of the diffeomorphism group $\mathrm{Diff}(M)$ of a compact manifold M. Milnor proves in [46] that the Lie algebra associated to $\mathrm{Diff}(M)$ is the space of vector field $\mathfrak{X}(M)$ on M, whose Lie bracket is the negative of the bracket product of vector fields. It turns out that an analogous result holds for the Lie algebra of the Lie group $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$. To understand the result we need the following facts:
A map of orbifolds $[\hat{\sigma}]$, which is a section of the tangent orbibundle is called an orbisection. With respect to an orbifold chart of Q, each orbisection induces a unique vector field on the chart domain, called its canonical lift. In particular each orbisection corresponds to a unique family of vector fields (cf. Section 4 for details). By construction, the local model for the Lie group $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ is the space of compactly supported orbisections $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$. We are now in a position to formulate the following result on the Lie algebra of the diffeomorphism group $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ (Theorem 6.3.1):

Theorem C *The Lie algebra of* $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ *is given by* $(\mathfrak{X}_{Orb}(Q)_c, [\cdot,\cdot])$. *Here the Lie bracket* $[\cdot,\cdot]$ *is defined as follows:*
For arbitrary $[\hat{\sigma}],[\hat{\tau}] \in \mathfrak{X}_{Orb}(Q)_c$, *their Lie bracket* $[[\hat{\sigma}],[\hat{\tau}]]$ *is the unique compactly supported orbisection whose canonical lift on an orbifold chart* (U,G,φ) *is the negative of the Lie bracket in* $\mathfrak{X}(U)$ *of their canonical lifts* σ_U *and* τ_U.

Finally we discuss regularity properties of the Lie group $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$. To this end recall the notion of regularity for Lie groups:
Let G be a Lie group modelled on a locally convex space, with identity element $\mathbf{1}$ and $r \in \mathbb{N}_0 \cup \{\infty\}$. We use the tangent map of the right translation $\rho_g\colon G \to G$, $x \mapsto xg$ by $g \in G$ to define $v.g := T_{\mathbf{1}}\rho_g(v) \in T_gG$ for $v \in T_{\mathbf{1}}(G) =: L(G)$. Following [16], [29] and [30], G is called C^r*-regular* if the initial value problem

$$\begin{cases} \eta'(t) &= \gamma(t).\eta(t) \\ \eta(0) &= \mathbf{1} \end{cases}$$

has a (necessarily unique) C^{r+1}-solution $\mathrm{Evol}(\gamma) := \eta\colon [0,1] \to G$ for each C^r-curve $\gamma\colon [0,1] \to L(G)$, and the map

$$\mathrm{evol}\colon C^r([0,1], L(G)) \to G, \quad \gamma \mapsto \mathrm{Evol}(\gamma)(1)$$

is smooth. If G is C^r-regular and $r \leq s$, then G is also C^s-regular. A C^∞-regular Lie group G is called *regular (in the sense of Milnor)* – a property first defined in [46]. Every finite dimensional Lie group is C^0-regular (cf. [50]). Several important results in infinite-dimensional Lie theory are only

available for regular Lie groups (see [46], [50], [29], cf. also [42] and the references in these works). We prove the following result (Theorem 6.4.11):

Theorem D *For a second countable orbifold, the Lie group* $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ *is* C^k*-regular for each* $k \in \mathbb{N}_0 \cup \{\infty\}$*. In particular this group is regular in the sense of Milnor.*

Notice that in general the orbifolds in the present paper are not assumed to be second countable. However our methods require second countability of the orbifold to prove that the evolution map evol is smooth. It is known that the approach outlined in the present paper may not be adapted to orbifolds which are not second countable. Hence we pose the following question:

Open Problem: Let $(Q,\mathcal{U})$ be a paracompact reduced orbifold which is not second countable. Is the Lie group $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ a C^r-regular Lie group for some $r \in \mathbb{N}_0 \cup \{\infty\}$?

The present article commences with a brief introduction to infinite dimensional calculus, orbifolds and their properties (Section 2). Our goal is to present a mostly self contained exposition of orbifolds and their morphisms. In particular Appendix E contains all necessary information about orbifold maps in the sense of [51]. However, the exposition avoids references to the groupoid morphisms after which these maps are modelled. The paper is organized as follows:
In Sections 3 and 4 classes of orbifold maps are discussed in the setting of [51]. These include orbifold diffeomorphisms, partitions of unity and sections of the tangent orbibundle. Afterwards, we consider Riemannian geometry on orbifolds and develop important tools employed in the proof of the central results of this paper. Finally the main results are contained in Section 6.
The less introductory material contained in the appendices should be taken on faith on a first reading. The presentation of this material in the text would have distracted from the main line of thought.

2. Preliminaries and Notation

2.0.1 Conventions In the present paper we work exclusively over the field of the real numbers $\mathbb{R}$. All topological spaces will be assumed to be Hausdorff. We write $\mathbb{N} := \{1, 2, \dots\}$, respectively $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

2.1. Differential calculus in infinite dimensional spaces

Basic references for differential calculus in locally convex spaces are [5,22,38]. Basic facts on infinite dimensional manifolds are compiled in Appendix C.1. For the readers convenience we recall various definitions and results, which may be looked up in [22, 23, 27]:

2.1.1 Definition Let E, F be locally convex spaces, $U \subseteq E$ be a subset, $f\colon U \to F$ a map and $r \in \mathbb{N}_0 \cup \{\infty\}$. If U is open, we say that f is C^r if the iterated directional derivatives

$$d^{(k)} f(x, y_1, \dots, y_k) := (D_{y_k} D_{y_{k-1}} \cdots D_{y_1} f)(x)$$

exist for all $k \in \mathbb{N}_0$ such that $k \leq r$, $x \in U$ and $y_1, \dots, y_k \in E$ and define continuous maps $d^{(k)} f\colon U \times E^k \to F$. If f is C^∞ it is also called smooth. We abbreviate $df := d^{(1)} f$.

2.1.2 Remark If E_1, E_2, F are locally convex spaces and $U \subseteq E_1, V \subseteq E_2$ open subsets together with a C^1-map $f\colon U \times V \to F$, then one may compute the *partial derivative partial derivative* $d_1 f$ with respect to E_1. These are defined as $d_1 f\colon U \times V \times E_1 \to F, d_1 f(x, y; z) := \lim_{t \to 0} t^{-1}(f(x + tz, y) - f(x, y))$. Analogously one defines the partial derivative $d_2 f$ with respect to E_2. The linearity of $df(x, y, \bullet)$ implies the so called Rule on Partial Differentials for $(x, y) \in U \times V, (h_1, h_2) \in E_1 \times E_2$:

$$df(x, y, h_1, h_2) = d_1 f(x, y, h_1) + d_2 f(x, y, h_2), \quad (x, y) \in U \times V, (h_1, h_2) \in E_1 \times E_2 \tag{2.1.1}$$

By [22, Lemma 1.10] $f\colon U \times V \to F$ is C^1 if and only if $d_1 f$ and $d_2 f$ exist and are continuous.

2.1.3 Definition (Differentials on non-open sets) (a) The set $U \subseteq E$ is called *locally convex* if every $x \in U$ has a convex neighborhood V in U.

(b) Let $U \subseteq E$ be a locally convex subset with dense interior. A mapping $f\colon U \to F$ is called C^r if $f_{|U^\circ}\colon U^\circ \to F$ is C^r and each of the maps $d^{(k)}(f_{|U^\circ})\colon U^\circ \times E^k \to F$ admits a (unique) continuous extension $d^{(k)} f\colon U \times E^k \to F$. If $U \subseteq \mathbb{R}$ and f is C^1, we obtain a continuous map $f'\colon U \to E, f'(x) := df(x)(1)$. We shall write $\frac{\partial}{\partial x} f(x) := f'(x)$. In particular if f is of class C^r, we define recursively $\frac{\partial^k}{\partial x^k} f(x) = (\frac{\partial^{k-1}}{\partial x^{k-1}} f)'(x)$ for $k \in \mathbb{N}_0$, such that $k \leq r$ where $f^{(0)} := f$.

Using these definitions one may define infinite dimensional manifolds as usual. We refer to Appendix C.1 for definitions and comments on the notation used. To discuss regularity properties of Lie groups, the notion of $C^{r,s}$-mappings is useful.

2.1.4 Definition ($C^{r,s}$-mappings) Let E_1, E_2 and F be locally convex spaces, U and V open subsets of E_1 and E_2 respectively and $r, s \in \mathbb{N}_0 \cup \{\infty\}$. A mapping $f\colon U \times V \to F$ is called a $C^{r,s}$-map, if for all $i, j \in \mathbb{N}_0$ such that $i \leq r, j \leq s$ holds, the iterated directional derivative

$$d^{(i,j)} f(x, y, w_1, \ldots, w_i, v_1, \ldots, v_j) := (D_{(w_i,0)} \cdots D_{(w_1,0)} D_{(0,v_j)} \cdots D_{(0,v_1)} f)(x, y)$$

exists for all $x \in U, y \in V, w_1, \ldots, w_i \in E_1, v_1, \ldots, v_j \in E_2$ and yield continuous mappings

$$\begin{aligned} d^{(i,j)} f\colon U \times V \times E_1^i \times E_2^j \to F, \\ (x, y, w_1, \ldots, w_i, v_1, \ldots, v_j) \mapsto (D_{(w_i,0)} \cdots D_{(w_1,0)} D_{(0,v_j)} \cdots D_{(0,v_1)} f)(x, y) \end{aligned}$$

Again this concept may be extended to maps on non-open domains with dense interior:

2.1.5 Definition Let E_1, E_2 and F be locally convex spaces, consider locally convex subsets with dense interior U of E_1 and V of E_2, and $r, s \in \mathbb{N}_0 \cup \{\infty\}$. We say that $f\colon U \times V \to F$ is a $C^{r,s}$-map, if $f|_{U^\circ \times V^\circ}\colon U^\circ \times V^\circ \to F$ is a $C^{r,s}$-map and for all $i, j \in \mathbb{N}_0$ such that $i \leq r, j \leq s$, the map

$$d^{(i,j)}(f|_{U^\circ \times V^\circ})\colon U^\circ \times V^\circ \times E_1^i \times E_2^j \to F$$

admits a continuous extension $d^{(i,j)} f\colon U \times V \times E_1^i \times E_2^j \to F$.

For further results and details on the calculus of $C^{r,s}$-maps we refer to [2].

2.1.6 Definition Let U, V be locally convex subsets with dense interior of locally convex spaces E_1, E_2 and F be a locally convex space. For $r, s \in \mathbb{N}_0 \cup \{\infty\}$ we define the spaces

$$\begin{aligned} C^r(U, F) &:= \{ f\colon U \to F | f \text{ is a mapping of class } C^r \} \\ C^{r,s}(U \times V, F) &:= \{ f\colon U \times V \to F | f \text{ is a mapping of class } C^{r,s} \}. \end{aligned}$$

Furthermore define $C(U, F) := C^0(U, F)$ and endow $C^r(U, F)$ with the compact open C^r-topology (see Section C.2)

2.2. Orbifold I: Moerdijks definition

In this section we introduce orbifolds as in the works of Moerdijk et al.. Our exposition follows [48] and for the readers convenience we restate the most important results:

2.2.1 Definition (Orbifold charts) Let Q be a topological space. An *orbifold chart* of dimension $n \geq 0$ is a triple (U, G, ϕ), where U is a connected open subset of $\mathbb{R}^n$, G is a finite subgroup of $\mathrm{Diff}(U)$ and $\phi\colon U \to U$ is an open map which induces a homeomorphism from the orbit space $U/G \to \phi(U)$. If (U, G, ϕ) is an orbifold chart on Q and S an open G-stable subset of U, the triple $(S, G_S, \phi_{|S})$ is again an orbifold chart called the *restriction of* (U, G, ϕ) on S. More generally, let (V, H, ψ) be another orbifold chart on Q. An *embedding* $\lambda\colon (V, H, \psi) \to (U, G, \phi)$ *of orbifold charts* is an embedding $\lambda\colon V \to U$ that $\phi \circ \lambda = \psi$.
We say that two orbifold charts (U, G, ϕ) and (V, H, ψ) of dimension n on Q are *compatible* if for any $z \in \phi(U) \cap \psi(V)$ there exist an orbifold chart (W, K, θ) on Q with $u \in \theta(W)$ and embeddings between orbifold charts $\lambda\colon (W, K, \theta) \to (U, G, \phi)$ and $\mu\colon (W, K, \theta) \to (V, H, \psi)$

The invariance of domain theorem (cf. [34, Theorem 2B.3]) asserts that any embedding of orbifold charts $\lambda\colon V \to U$ has an open image, since U, V are n-dimensional manifolds.

2.2.2 Proposition ([48, Proposition 2.12]) *Let Q be a topological space.*

(a) *For any embedding $\lambda\colon (V, H, \psi) \to (U, G, \phi)$ between orbifold charts on Q, the image $\lambda(V)$ is a G-stable open subset of U, and there is a unique isomorphism $\overline{\lambda}\colon H \to G_{\lambda(V)} \leq G$ for which $\lambda(hx) = \overline{\lambda}(h)\lambda(x)$.*
(b) *The composition of two embeddings between orbifold charts is an embedding between orbifold charts.*
(c) *For any orbifold chart (U, G, ϕ), any diffeomorphism $g \in G$ is an embedding of (U, G, ϕ) into itself, and $\overline{g}(g') = gg'g^{-1}$.*
(d) *If $\lambda, \mu\colon (V, H, \phi) \to (U, G, \phi)$ are two embeddings between the same orbifold charts, there exists a unique $g \in G$ with $\lambda = g \circ \mu$.*

2.2.3 Definition (Orbifold I) An *orbifold atlas* of dimension n of a topological space Q is a collection of pairwise compatible orbifold charts

$$\mathcal{U} := \{\, (U_i, G_i, \phi_i)\,\}_{i \in I}$$

of dimension n on Q such that $\bigcup_{i \in I} \phi_i(U_i) = Q$. Two orbifold atlases of Q are equivalent if their union is an orbifold atlas. As for manifolds we may join all equivalent atlases to obtain a maximal orbifold atlas. An *orbifold* of dimension n is a pair $\mathcal{O} := (Q_{\mathcal{O}}, \mathcal{U}_{\mathcal{O}})$, where $Q_{\mathcal{O}}$ is a paracompact Hausdorff topological space and $\mathcal{U}_{\mathcal{O}}$ a maximal orbifold atlas of dimension n on $Q_{\mathcal{O}}$.
In general we shall not assume that $Q_{\mathcal{O}}$ is a second countable topological space. For most of our results, second countability is not necessary, thus we chose to omit it here (contrary to [48]).

2.3. Orbifold II: Haefligers definition

We introduce an equivalent definition of Orbifolds as outlined in [32] (cf. [48, Proposition 2.13] for a proof of the equivalence).

2.3.1 Definition (Orbifold II, [32]) Let Q be a paracompact Hausdorff topological space.

(a) Let $n \in \mathbb{N}_0$. A *(reduced) orbifold chart* of dimension n on Q is a triple (V, G, φ) where V is a connected paracompact n-dimensional manifold without boundary, G is a finite subgroup of $\mathrm{Diff}(V)$, and $\varphi\colon V \to Q$ is a map with open image $\varphi(V)$ that induces a homeomorphism from V/G to $\varphi(V)$. In this case (V, G, φ) is said to *uniformize* $\varphi(V)$.

(b) Two reduced orbifold charts (V, G, φ), (W, H, ψ) on Q are called *compatible* if for each pair $(x, y) \in V \times W$ with $\varphi(x) = \psi(y)$ there are open connected neighbourhoods V_x of x and W_y of y and a diffeomorphism $h\colon V_x \to W_y$ such that $\psi \circ h = \varphi|_{V_x}$. The map h is called a *change of charts*.

(c) A *reduced orbifold atlas* of dimension n on Q is a collection of pairwise compatible reduced orbifold charts
$$\mathcal{V} := \{\, (V_i, G_i, \varphi_i) | i \in I \,\}$$
of dimension n on Q such that $\bigcup_{i\in I} \varphi_i(V_i) = Q$.

(d) Two reduced orbifold atlases are *equivalent* if their union is a reduced orbifold atlas.

(e) A *reduced orbifold structure* of dimension n on Q is a (with respect to inclusion) maximal reduced orbifold atlas of dimension n on Q, or equivalently, an equivalence class of reduced orbifold atlases of dimension n on Q.

(f) A *reduced orbifold* of dimension n is a pair $(Q, \mathcal{U})$ where $\mathcal{U}$ is a reduced orbifold structure of dimension n on Q.

2.3.2 Remark (a) The term "reduced" refers to the requirement that for each reduced orbifold chart (V, G, φ) in $\mathcal{U}$ the group G be a subgroup of $\mathrm{Diff}(V)$. Hence the action of G on V is effective. As [48, Proposition 2.13] shows, the definitions Orbifold I and Orbifold II are equivalent. We will only consider reduced orbifolds (and maps between them) so to shorten our notation, we will drop the term "reduced" in the remainder of the paper. A "reduced" orbifold will thus simply be called an orbifold.

(b) We will occasionally refer to the dimension of an orbifold as defined in 2.3.1 as the *orbifold dimension*. We shall prove later that as in the case of a manifold, the orbifold dimension is an invariant of the orbifold. More explitely two orbifolds may only be diffeomorphic to each other if they have the same orbifold dimension. We postpone these considerations until we are ready to define morphisms of orbifolds.

(c) In general maps of orbifolds (see Section E) only admit local lifts in certain orbifold atlases contained in the maximal atlas of the orbifold $(Q, \mathcal{U})$. Therefore we introduce the convention: An atlas $\mathcal{V}$ contained in the maximal atlas $\mathcal{U}$ of the orbifold $(Q, \mathcal{U})$ will be called *representative of* $\mathcal{U}$.

2.4. The topology of the base space of an Orbifold

In this section we compile several facts about orbifolds which are well known in the literature (cf. [1, 6, 14, 48]). Nevertheless we give full proofs for the readers convenience:

2.4.1 Lemma *For any orbifold $(Q,\mathcal{U})$, the family of open subsets $\left\{ \tilde{V} := \pi(V) \middle| (V,G,\pi) \in \mathcal{U} \right\}$ is a base for the topology on Q.*

Proof. Let $p \in Q$ and $p \in U \subseteq Q$ an open set. Choose an orbifold chart $(V,G,\pi) \in \mathcal{U}$, such that $p \in \tilde{V} = \pi(V)$. The map π is given by the composition of the quotient map onto the orbit space with a homeomorphism onto an open set. Hence Lemma B.1.4 shows that π is continuous and open. The set $\pi^{-1}(U)$ is an open subset of V containing some element $\hat{p} \in \pi^{-1}(p)$. By Lemma B.1.3 we may choose a $G_{\hat{p}}$-invariant open set S such that $\hat{p} \in S \subseteq \pi^{-1}(U)$ and $(S, G_{\hat{p}, \pi_{|S}})$ is an orbifold chart. By construction $p \in \pi(S) \subseteq U$ proving our claim. □

To analyse the structure of the base space we need a well known fact from topology:

2.4.2 Proposition *If X is a Hausdorff space that is locally compact and paracompact, then each component of X is σ-compact. If in addition X is locally metrizable, then X is metrizable and every component has a countable basis of the topology.*

Proof. By [19, XI. Thm. 7.3] each component is σ-compact. The space X is paracompact, locally metrizable and hausdorff, hence we may choose a locally finite closed cover consisting of metrizable subspaces. Then X is metrizable by [20, Thm. 4.4.19]). Each component C is Lindelöf, by [19, XI. Thm. 7.2]. We deduce from [20, Corollary 4.1.16] that C is second countable. □

2.4.3 Proposition *Let $(Q,\mathcal{U})$ be an orbifold, the topological space Q has the following properties:*

(a) *Q is a locally compact hausdorff space.*
(b) *Q is connected if and only if Q is path connected*
(c) *Q is metrizable.*
(d) *Every connected component C of Q is open, σ-compact and second countable*

We remark, that Q is not necessarily second countable.

Proof. (a) Q is Hausdorff by definition of an orbifold. Clearly being a locally compact space is a local condition, i.e. may be checked within $\pi(U)$, where $(U,G,\pi) \in \mathcal{U}$ is an arbitrary orbifold chart. Lemma B.1.4 shows that $\pi(U)$ is a locally compact Hausdorff space, since every finite dimensional Hausdorff manifold U is such a space.

(b) The quotient map onto the orbit space is continuous (Lemma B.1.4) and manifolds are locally path-connected. Thus Q is locally path connected, whence the assertion follows from general topology [19, V. Theorem 5.5].

(c) For every chart $(U, G, \pi) \in \mathcal{U}$ the group $G \subseteq \mathrm{Diff}(U)$ is finite. The preimage $\pi^{-1}(y)$ of any $y \in \pi(U)$ is finite (and thus compact). The manifold U is locally metrizable (since every chart is a homeomorphism) and a paracompact locally compact Hausdorff space. By Proposition 2.4.2, U is metrizable. The quotient map onto an orbit space is a closed-and-open map by Lemma B.1.4. Since metrizability is an invariant of closed-and-open maps by [20, Theorem 4.2.13], the space Q is locally metrizable. Summing up Q is a locally metrizable, locally compact and paracompact Hausdorff space. Again by Proposition 2.4.2 the metrizability of Q follows.

(d) Q is locally path connected, which implies the openness of C by [19, V. 5.4]. We already know that Q is a Hausdorff space which is paracompact and locally compact. Every component of Q is then σ-compact and second countable by Proposition 2.4.2.
To prove the last remark, consider the following counterexample: Let $(Q,\mathcal{U})$ be an arbitrary orbifold modelled on a topological space Q and a set I with cardinality at least $\aleph_1$. Construct the orbifold Q^I by defining the topological space $X_{\mathcal{O}^I} := X^I_{\mathcal{O}} = \coprod_{i \in I} X_{\mathcal{O}}$ as the disjoint union of copies of $X_{\mathcal{O}}$ and the orbifold charts on every copy of $X_{\mathcal{O}}$ as copies of charts in $\mathcal{O}$. Then $X^I_{\mathcal{O}}$ is *not second countable*, even if $X_{\mathcal{O}}$ is.

$\square$

2.5. Local groups and the singular locus

Let $(Q,\mathcal{U})$ be an orbifold, $(U, G, \pi) \in \mathcal{U}$ an orbifold chart of Q and $x \in U$. Let $z := \pi(x)$. We deduce from [48, Lemma 2.10] that the differential at x induces a faithful representation of G_x in $\mathrm{Gl}(n, \mathbb{R})$. The corresponding finite subgroup of $\mathrm{Gl}(n, \mathbb{R})$ will be called TG_x. Since $G_{gx} = gG_xg^{-1}$ for any $g \in G$, points in the same orbit of G have isotropy groups in the same conjugacy class of $\mathrm{Gl}(n, \mathbb{R})$. In particular TG_x and TG_{gx} are in the same conjugacy class of $\mathrm{Gl}(n, \mathbb{R})$. Let $\lambda\colon (V, H, \psi) \to (U, G, \pi)$ be an embedding of orbifold charts and $y \in V$ with $\lambda(y) = x$. Observe that $\overline{\lambda}(H_y) = G_x$ by Proposition 2.2.2 and thus

$$TG_x = T_y\lambda TH_y(T_y\lambda)^{-1}$$

Thus the conjugacy class of TG_x depends only on the point z and not on the choice of the orbifold chart (U, G, π) on Q or of x. Hence the following definition is justified.

2.5.1 Definition (local group) Let $(Q,\mathcal{U})$ be an orbifold. For every $z \in Q$, by the above there is a group $\Gamma_z(Q) \subseteq \mathrm{Gl}(n, \mathbb{R})$ which is unique up to conjugation in $\mathrm{Gl}(n, \mathbb{R})$. We call $\Gamma_z(Q)$ the *local group* of z. In the literature $\Gamma_z(Q)$ is also called the *isotropy group* of z. We avoid this and reserve "isotropy group" for the subgroup of a group acting on a manifold, which fixes a given point.

The singularities, i.e. points with non-trivial local group, generate a structure which distinguishes a non-trivial orbifold from a manifold. We claimed that orbifolds are manifolds with "mild singulari-

ties". To emphasise this point we shall investigate the singular locus (i.e. the set of all singularities). As a consequence of Newmans Theorem B.2.1, the singular locus is a nowhere dense closed subset of the base space of an orbifold. In other words, the topological base space of an orbifold contains an open and dense manifold. A proof for this result is given in the rest of this section:

2.5.2 Definition (Singular locus) Let $(Q,\mathcal{U})$ be an orbifold. The *singular locus* of Q is the subset

$$\Sigma_Q := \{ z \in Q | \Gamma_z(Q) \neq \{ 1 \} \}$$

In a chart (U, G, π) one has $\Sigma_Q \cap \pi(U) = \pi(\Sigma_G)$, where Σ_G is the set points with non trivial isotropy subgroup with respect to the action of G. An element $x \in Q$ is called *singular point* if $x \in \Sigma_Q$ and x is called *non-singular* if $x \notin \Sigma_Q$.
Since there are different orbifold structures on the same topological space, occasionally we have to indicate which one is meant. In these cases we shall write $\Gamma_z(Q,\mathcal{U})$ resp. $\Sigma_{(Q,\mathcal{U})}$, to avoid confusion.

2.5.3 Proposition (Newman, Thurston) *The singular locus Σ_Q of an orbifold $(Q,\mathcal{U})$ is a closed set with empty interior.*

Proof. Let (U, G, π) be any chart at some point $p \in Q$. By definition $\Sigma_Q \cap \pi(U)$ is the image of Σ_G. As $G \subseteq \mathrm{Diff}(U)$ is finite, we deduce from Newmans theorem B.2.1 that the set $\mathcal{N}_U$ of non singular points in U is open and dense. Lemma B.1.4 shows that the quotient map π onto the orbit space is open, hence

$$\Sigma_Q = Q \setminus \bigcup_{(U,G,\pi)\in\mathcal{U}} \pi(\mathcal{N}_U)$$

is a closed set. Each connected component $C \subseteq Q$ is open by Proposition 2.4.3. To prove $\Sigma_Q^\circ = \emptyset$ we check that $\Sigma_Q \cap C$ has empty interior for every connected component C of Q. The component C is second countable by Proposition 2.4.3. Hence we may choose countably many charts $(U_i, G_i, \pi_i) \in \mathcal{U},\ i \in \mathbb{N}$ whose images are contained in C and cover C. Proposition 2.4.3 shows that C is a locally compact hausdorff space, whence Čech-complete (cf. [20, p. 196]. The set $\Sigma_Q \cap C = \bigcup_{i=1}^{\infty} \pi(U_i) \cap \Sigma_Q$ has empty interior by the Baire category theorem [20, 3.9.3]. $\square$

2.6. Orbifold atlases with special properties

In this section we construct orbifold atlases with properties. These atlases are needed later on, to construct chart for the diffeomorphism group of an orbifold.

2.6.1 Definition Let $(Q,\mathcal{U})$ be an orbifold and $\mathcal{V}$ a representative of $\mathcal{U}$. We say that another representative $\mathcal{W}$ of $\mathcal{U}$ *refines* $\mathcal{V}$ (or is a *refinement of the atlas* $\mathcal{V}$) if for every chart $(W,G,\psi) \in \mathcal{W}$, there is a chart $(V,H,\pi) \in \mathcal{V}$ and an open embedding of orbifold charts $\lambda_{W,V} : (W,G,\psi) \to (V,H,\pi)$. Given another representative $\mathcal{V}'$ of $\mathcal{U}$, we say that $\mathcal{W}$ is a *common refinement of* $\mathcal{V}$ *and* $\mathcal{V}'$, if $\mathcal{W}$ refines $\mathcal{V}$ and $\mathcal{W}$ refines $\mathcal{V}'$.

2.6.2 Lemma *For an orbifold $(Q,\mathcal{U})$ and two arbitrary representatives $\mathcal{V},\mathcal{V}'$ of $\mathcal{U}$, there exists a common refinement $\mathcal{W}$ of $\mathcal{V}$ and $\mathcal{V}'$.*

Proof. Every $x \in Q$ is contained in the image of some charts $(V,G,\pi) \in \mathcal{V}$ and $(V',G',\pi') \in \mathcal{V}'$. Since both definitions of orbifolds are equivalent, by Definition 2.2.1 there is $(W,H,\psi) \in \mathcal{U}$ together with open embeddings $\lambda\colon (W,H,\psi) \to (V,G,\pi)$ and $\mu\colon (W,H,\psi) \to (V',G',\pi')$, such that $x \in \psi(W)$. For each $x \in Q$ we may choose a chart with these properties and the charts chosen in this way form an atlas and a common refinement of $\mathcal{V}$ and $\mathcal{V}'$ by construction. □

2.6.3 Lemma *Let $(Q,\mathcal{U})$ be an orbifold, for any representative $\mathcal{V}$ of $\mathcal{U}$, consider the families of orbifold charts*

$$\mathcal{U} \Subset \mathcal{V} := \{\, (U,H,\phi) \in \mathcal{U} | \exists \lambda_{U,V} : (U,H,\phi) \to (V,G,\psi) \text{ embedding, for some } (V,G,\psi) \in \mathcal{V} \,\}$$
$$\mathcal{U} \sqsubset \mathcal{V} := \left\{ (U,H,\phi) \in \mathcal{U} \Subset \mathcal{V} \middle| \overline{\phi(U)} \subseteq \psi(V) \text{ and } \overline{\lambda_{U,V}(U)} \subseteq V \text{ is compact} \right\}$$

Then the families of open sets $\{\, \phi(U) | (U,H,\phi) \in \mathcal{U} \Subset \mathcal{V} \,\}$ and $\{\, \phi(U) | (U,H,\phi) \in \mathcal{U} \sqsubset \mathcal{V} \,\}$ are bases for the topology on Q.

Proof. Consider an arbitrary open set $\Omega \subseteq Q$ and some point $x \in \Omega$. The family $\mathcal{V}$ is an atlas and thus, there is some chart $(V,G,\psi) \in \mathcal{V}$ with $x \in \operatorname{Im}\psi$. We already know that the open sets $\{\, \phi(U) | (U,H,\phi) \in \mathcal{U} \,\}$ form a base of the topology by Lemma 2.4.1. Thus we may choose some chart $(U,H,\phi) \in \mathcal{U}$, such that $x \in \operatorname{Im}\phi \subseteq \Omega$. Choose an arbitrary preimage $z \in \phi^{-1}(x) \subseteq U$. The atlas $\mathcal{V}$ is a representative of $\mathcal{U}$ and by Definition 2.2.1 there is $(W,K,\pi) \in \mathcal{U}$ with embeddings of orbifold charts $\lambda\colon (W,K,\pi) \to (V,G,\psi)$ and $\theta\colon (W,K,\pi) \to (U,H,\phi)$ such that $z \in \operatorname{Im}\theta$. By construction, (W,K,π) belongs to $\mathcal{U} \Subset \mathcal{V}$. The embedding θ assures $x \in \operatorname{Im}\pi = \operatorname{Im}\phi \circ \theta \subseteq \operatorname{Im}\phi \subseteq \Omega$. Hence each $x \in \Omega$ is contained in $\pi(W) \subseteq \Omega$ for some $(W,K,\pi) \in \mathcal{U} \Subset \mathcal{V}$. As Ω was arbitrary, this proves that the images of $\mathcal{U} \Subset \mathcal{V}$ form a base of the topology.
To prove the second statement observe that for (W,K,π), the topological space W is locally compact. Hence there is a relatively compact K-stable neighborhood $x \in W' \subseteq W$. This neighborhood induces an orbifold chart $(W',K_{W'},\pi_{|W'}) \in \mathcal{U}$. Combining continuity of π and compactness of $\overline{W'}$ the set $\pi(W') \subseteq \pi(\overline{W'}) \subseteq \operatorname{Im}\pi \subseteq \Omega$ is relatively compact. As $(W,K,\pi) \in U \Subset \mathcal{V}$, there is an open

embedding of orbifold charts $\lambda\colon (W,K,\pi) \to (V,G,\psi)$. Thus $\overline{\pi(W')} \subseteq \operatorname{Im}\psi$ holds and $\lambda_{|W'}$ is an open embedding of orbifold charts by Proposition 2.2.2 (b). Using compactness of $\overline{W'}$, we conclude from standard topology (cf. [20, Corollary 3.1.11.]) that $\overline{\lambda(W')} = \lambda(\overline{W'})$ is compact. Therefore $\lambda(W')$ is relatively compact. Again since x and Ω were arbitrary, the second statement follows. □

2.6.4 Definition Let $(Q,\mathcal{U})$ be an orbifold. An orbifold atlas $\mathcal{V} := \{\, (V_i, G_i, \pi_i) | i \in I \,\}$ of $(Q,\mathcal{U})$, is called *locally finite orbifold atlas* if the family $\{\, \pi_i(V_i) | i \in I \,\}$ is a locally finite family of open sets.

2.6.5 Lemma *Let $(Q,\mathcal{U})$ be an orbifold.*

(a) *There is a locally finite representative $\mathcal{V}$ of $\mathcal{U}$.*
(b) *For each representative $\mathcal{W}$ of $\mathcal{U}$, there is a locally finite representative $\mathcal{W}'$, which refines $\mathcal{W}$.*
(c) *The refinement $\mathcal{W}'$ in (b) may be chosen with the following property:*
For each $(U,G,\psi) \in \mathcal{W}'$, there are $(V,H,\varphi) \in \mathcal{W}$ and an open embedding $\lambda_{U,V}$ of orbifold charts, such that $\overline{\tilde{U}} \subseteq \tilde{V}$ is a compact set and $\overline{\lambda_{U,V}(U)} \subseteq V$ is compact.

Taking identifications, without loss of generality $\lambda_{U,V}$ is just the canonical inclusion (of sets) and G is a subgroup of H.

Proof. (a) The topological space Q is a locally compact Hausdorff space. For each $q \in Q$ pick a compact neighborhood U_q of q. Then $(U_q^\circ)_{q\in Q}$ is an open covering of Q. By paracompactness of Q there is a locally finite open refinement $(W_j)_{j\in J}$ of $(U_q^\circ)_{q\in Q}$. Note that every $\overline{W_j}$ is compact. By [20, Lemma 5.1.6], there exists a shrinking $(O_j)_{j\in J}$ of $(W_j)_{j\in J}$, that is an open covering of Q such that $\overline{O_j} \subseteq W_j$ for each $j \in J$. The family of uniformized subsets of Q, form a basis of the topology by 2.4.1. Thus for each $j \in J$, the compact set $\overline{O_j}$ is covered by finitely many uniformized sets which are contained in W_j, say $\overline{O_j} \subseteq \bigcup_{k=1}^n B_{j,k}$. Since the family $(W_j)_{j\in J}$ is locally finite,

$$\{\, B_{j,k} | j \in J,\ k = 1, \ldots, n \,\}$$

is a locally finite open covering of Q by uniformized subsets. The corresponding atlas $\mathcal{V}$ is thus locally finite.

(b) and (c) We may argue as in (a), but replace the set of all uniformized subsets of Q by the set of all uniformized subsets, which are images of $\mathcal{U} \Subset \mathcal{W}$ (resp. images of $\mathcal{U} \sqsubset \mathcal{W}$ for (c)). Since Lemma 2.6.3 assures that these sets of images are bases of the topology, no further changes in the proof are needed. For the last statement identify U and $\lambda_{U,V}(U)$ resp. G with $\overline{\lambda}(G)$.

□

2.6.6 Lemma *Let $(Q,\mathcal{U})$ be an orbifold and $\mathcal{W}$ a locally finite orbifold atlas, such that for each $(V,H,\varphi) \in \mathcal{W}$ the uniformized subset $\varphi(V)$ is relatively compact. Consider a refinement $\mathcal{W}'$ as in Lemma 2.6.5 (c) indexed by a set I. There exists a map $\alpha\colon I \to \mathcal{W}$, which associates to each i a chart $(V_{\alpha(i)}, H_{\alpha(i)}, \varphi_{\alpha(i)})$ into which (U_i, G_i, ψ_i) embedds (as an orbifold chart) via an embedding of sets $U_i \to V_{\alpha(i)}$. Furthermore $I_V := \alpha^{-1}(V,H,\varphi) \subseteq I$ is finite for each $(V,H,\varphi) \in \mathcal{W}$.*

Proof. Lemma 2.6.5 (c) assures that for each $i \in I$, there is at least one chart in $\mathcal{W}$, such that (U_i, G_i, ψ_i) embedds into this chart via the inclusion of sets. Choose a chart $(V_{\alpha(i)}, H_{\alpha(i)}, \varphi_{\alpha(i)})$ such that $\overline{U_i} \subseteq V_{\alpha(i)}$ is compact, $G_i \subseteq H_{\alpha(i)}$ and $\psi_i = \varphi_{\alpha(i)|U_i}$ holds. We obtain a map $\alpha\colon I \to \mathcal{W}$ with the desired properties. For each $(V, H, \varphi) \in \mathcal{W}$, the uniformized subset $\varphi(V)$ is relatively compact. Since $\mathcal{W}'$ is locally finite there is only a finite subset of I, such that $\psi_i(U_i) \cap \overline{\varphi(V)} \neq \emptyset$. Therefore $I_V := \alpha^{-1}(V, H, \varphi)$ is finite for each $(V, H, \varphi) \in \mathcal{W}$. □

2.6.7 Proposition *Let $(Q, \mathcal{U})$ be an orbifold, $\mathcal{V} \subseteq \mathcal{U}$ some orbifold atlas and D a discrete subset of Q. There exist locally finite atlases $\mathcal{A} := \{ (U_i, G_i, \psi_i) | i \in I \}, \mathcal{B} := \{ (W_j, H_j, \varphi_j) | j \in J \} \subseteq \mathcal{U}$, such that each the following conditions are satisfied:*

(a) *the charts in $\mathcal{A}, \mathcal{B}$ are relatively compact, i.e. if (U, G, ψ) is such a chart, the set $\overline{\psi(U)}$ is a compact subset of Q,*
(b) *the atlas $\mathcal{A}$ refines $\mathcal{B}$ and $\mathcal{B}$ refines $\mathcal{V}$ as in Lemma 2.6.5 (c),*
(c) *For $z \in D$, there are unique $i_z \in I, j_z \in J$ with $z \in \psi_i(V_i)$ (resp. $z \in \varphi_j(U_j)$) iff $i = i_z$ (resp. $j = j_z$),*
(d) *If Q is σ-compact and D is countable, the sets I, J are countable*

Proof. The space Q is a metrizable locally compact space by Proposition 2.4.3. Let d be the metric which induces the topology on Q. Since D is discrete, for each $z \in D$ there is a unique $r_z := \sup \{ r > 0 | B_r(z) \cap D = \{ z \} \}$ (where $B_r(z)$ is the metric ball of radius r centered at z). We claim that the balls $B_{\frac{r_z}{3}}(z), z \in D$ are disjoint. Assume that this were not the case and let $y, z \in D$ with $y \neq z$ and $x \in B_{\frac{r_y}{3}}(y) \cap B_{\frac{r_z}{3}}(z)$. From the triangle inequality we deduce the contradiction

$$d(y, z) \leq d(y, x) + d(x, z) = \tfrac{1}{3}(r_y + r_z) \leq \tfrac{2}{3} d(x, y).$$

As Q is locally compact, we may choose for each $z \in D$ a compact neighborhood $L_{1,z} \subseteq B_{\frac{r_z}{3}}(z)$. Each pair of such neighborhoods is disjoint. By Lemma 2.6.3 for each z there is a pair of relatively compact orbifold charts $(U_z, G_z, \psi_z), (W_z, H_z, \varphi_z) \in \mathcal{U} \sqsubset \mathcal{V}$ such that $\overline{U_z} \subseteq W_z$ and $z \in \psi_z(U_z) \subseteq \overline{\varphi_z(W_z)} \subseteq L_{1,z}^\circ$ holds. Furthermore the inclusion of sets induces an embedding of orbifold charts. Again by local compactness, we may choose for each z compact neighborhoods $z \in L_{2,z} \subsetneq \psi_z(U_z)$. The set $L_{2,z}$ is contained in $L_{1,z}$. Since the sets $L_{1,z}$ are disjoint, the family $\{ L_{2,z} \}_{z \in D}$ is locally finite. The set $L := \bigcup_{z \in D} L_{2,z}$ is thus closed by [20, Cor. 1.1.12] and we may consider the open subset $Q' := Q \setminus L$. Now Q' is locally compact and as Q is metrizable by Proposition 2.4.3, the subspace Q' is paracompact. Furthermore the set of charts $\mathcal{R} := \{ (V, H, \pi) \in \mathcal{U} \sqsubset \mathcal{V} | \pi(V) \subseteq Q' \}$ is a basis for the topology on Q'. Using an argument analogous to Lemma 2.6.5 (c) there is a locally finite orbifold atlas $\mathcal{B}' \subseteq \mathcal{R}$ for Q', such that each chart $(W, H, \varphi) \in \mathcal{B}'$ is relatively compact and embedds into some member of $\mathcal{V}$ as in Lemma 2.6.5 (c). Another application of Lemma 2.6.5 (c) yields a locally finite orbifold atlas $\mathcal{A}' := \{ (U_i, G_i, \psi_i) \}_{i \in I'}$ for Q', which is a refinement of $\mathcal{B}'$ with properties as in 2.6.5 (c). Notice that by construction none of the charts in $\mathcal{B}'$ and $\mathcal{A}'$ contain elements of D.

For each $z \in D$ the set $L_z := L_{1,z} \cap Q \setminus \psi_z(U_z) \subseteq Q'$ is compact. The atlases $\mathcal{B}', \mathcal{A}'$ are locally finite and thus there are finite subsets $J'_z \subseteq J'$ (resp. $I'_z \subseteq I'$) such that $\varphi_j(W_j) \cap L_z \neq \emptyset$ iff $j \in J'_z$ (resp.

$\psi_i(U_i) \cap L_z \neq \emptyset$ iff $i \in I'_z$). Assume that P is the image of an orbifold chart in $\mathcal{A}'$ resp. $\mathcal{B}'$ which is contained in

$$O := Q \setminus \bigcup_{z \in D} L_z = (\bigcup_{z \in D} \psi_z(U_z)) \cup Q \setminus \bigcup_{z \in D} L_{1,z}.$$

As each $L_{1,z}$ is a closed set and the family $\{ L_{1,z} \}_{z \in D}$ is disjoint, whence locally finite, the union of the sets $L_{1,z}$ is closed by [20, Cor. 1.1.12]. Therefore O is an open set and by construction

$$P = \left(\bigcup_{z \in D} \psi_z(U_z) \cap P \right) \cup \left(P \cap Q \setminus \bigcup_{z \in D} L_{1,z} \right)$$

is a disjoint union of two open sets. As orbifold charts are connected, we deduce that their images are located as follows:
Either the image is contained in $Q \setminus \bigcup_{z \in D} L_{1,z}$, or it intersects at least one of the $L_z, z \in D$, or it is contained in $\bigcup_{z \in D} \psi_z(U_z)$. Discarding the charts, whose image is contained in $\bigcup_{z \in D} \psi_z(U_z)$, we obtain subsets

$$J'' := \bigcup_{z \in D} J'_z \cup \left\{ j \in J \middle| \varphi_j(W_j) \cap \bigcup_{z \in D} L_{1,z} = \emptyset \right\}, \quad I'' := \bigcup_{z \in D} I'_z \cup \left\{ i \in I \middle| \psi_i(U_i) \cap \bigcup_{z \in D} L_{1,z} = \emptyset \right\}$$

of I', J', such that the families $\mathcal{B}'' := \{ (W_j, H_j, \varphi_j) | j \in J'' \}$ and $\mathcal{A}'' := \{ (U_i, G_i, \psi_i) | i \in I'' \}$ cover $Q \setminus \bigcup_{z \in D} \psi_z(U_z)$. It is easy to check that the construction forces $\mathcal{A}''$ to be a a refinement of $\mathcal{B}''$ with the properties described in Lemma 2.6.5 (c).
Set $I := I'' \coprod D$ and $J := J'' \coprod D$. The sets index orbifold atlases $\mathcal{A} := \mathcal{A}'' \cup \{ (U_z, G_z, \psi_z) | z \in D \}$ resp. $\mathcal{B} := \mathcal{B}'' \cup \{ (W_z, H_z, \varphi_z) | z \in D \}$. By construction $\mathcal{A}$ is a refinement of $\mathcal{B}$ with the properties of Lemma 2.6.5 (c).
It remains to prove that $\mathcal{A}$ and $\mathcal{B}$ are locally finite: As $\mathcal{A}''$ and $\mathcal{B}'''$ are locally finite atlases, it suffices to check the following condition: For each $z \in D$, only finitely many charts in $\mathcal{A}''$ (resp. $\mathcal{B}''$) intersect the image of (U_z, G_z, ψ_z) (resp. (W_z, H_z, φ_z)).
For each $z \in D$ the charts indexed by z are contained in $L_{1,z}$ and by construction only a finite number of charts in $\mathcal{A}''$ (resp. $\mathcal{B}''$) intersect $L_{1,z}$. Thus at most finitely many images of charts in $\mathcal{A}$ resp. $\mathcal{B}$ intersect a given $L_{1,z}^\circ, z \in D$, whence $\mathcal{A}$ and $\mathcal{B}$ are locally finite.
We claim that if Q is σ-compact, the atlases $\mathcal{A}'$ and $\mathcal{B}'$ are indexed by countable sets. If this were true and D is countable, the construction above yields countable atlases since I'', J'' are countable as subsets of the countable sets I' and J'. Hence the assertion (d) holds.
Proof of the claim: If Q is σ-compact, then it contains at most countably many connected components. Thus by Proposition 2.4.3 (d), Q is second-countable as disjoint union of at most countably many second countable spaces. By [19, Theorem 6.2 (2)] each subspace of Q is second countable, whence a Lindelöf space. As open subsets of locally compact spaces are locally compact, we conclude that Q' is σ-compact by [19, Theorem 7.2]. By definition there are compact sets $C_n, n \in \mathbb{N}$ such that $Q' = \bigcup_{n \in \mathbb{N}} C_n$. The atlases $\mathcal{A}'$ and $\mathcal{B}'$ are locally finite, whence each compact set C_n intersects only finitely many charts in $\mathcal{B}'$ and $\mathcal{B}'$. Clearly this forces the atlases to be indexed by countable sets I' resp. J'. □

The following technical Lemma will allow us to control the local behavior of sections into the tangent orbifold.

2.6.8 Lemma *Let $(Q,\mathcal{U})$ be an orbifold and $\mathcal{W} = \{ (V_i, H_i, \varphi_i) | i \in I \}$ be a locally finite orbifold atlas. For each $i \in I$ let there be a compact subset $K_i \subseteq V_i$. Then for each $i \in I$ there is an open cover $\{ Z_i^k \}_{1 \leq k \leq n_i}$ of $K_i \subseteq V_i$ such that*

(a) *the sets Z_i^k are H_i-stable for $i \in I$, $1 \leq k \leq n_i$,*
(b) *for each $j \in I$ with $Z_i^k \cap K_i \cap \varphi_i^{-1}\varphi_j(K_j) \neq \emptyset$ there is an embedding of orbifold charts $\lambda_{ij}^k \colon Z_i^k \to V_j$ with $\lambda_{ij}^k(Z_i^k \cap K_i \cap \varphi_i^{-1}\varphi_j(K_j)) \subseteq H_j.K_j$.*
(c) *The covering $\{ Z_i^k \}_{1 \leq k \leq n_i}$ may be chosen such that for each $i \in I, 1 \leq k \leq N_i$ there is a H_i-stable set open set $\hat{Z}_i^k$, such that $\overline{Z_i^k}$ is a compact set, contained in $\hat{Z}_i^k$ and each embedding λ_{ij}^k is the restriction of an embedding on $\hat{Z}_i^k$.*

Proof. The set $\tilde{K}_i := \varphi_i(K_i)$ is compact and since $\mathcal{W}$ is locally finite, there is a finite subset $\mathcal{F}_i$ of $\mathcal{W}$, such that $\tilde{K}_i \cap \varphi(V) \neq \emptyset$ if and only if $(V, H, \varphi) \in \mathcal{F}_i$. In particular there is a finite set $j \in J_i$, such that $\tilde{K}_{ij} := \tilde{K}_i \cap \varphi_j(K_j) \neq \emptyset$ iff $j \in J_i$. The compact sets $\tilde{K}_{ij}$ are contained in $\tilde{V}_i$. By Lemma B.1.4 the quotient map to an orbit space is a proper map, whereas the following sets are compact

$$\begin{aligned} K_{ij} :=& K_i \cap \varphi_i^{-1}(\tilde{K}_{ij}) = K_i \cap \varphi_i^{-1}(\varphi_i(K_i) \cap \varphi_j(K_j)) = K_i \cap \varphi_i^{-1}\varphi_i(K_i) \cap \varphi_i^{-1}\varphi_j(K_j) \\ =& K_i \cap H_i.K_i \cap \varphi_i^{-1}\varphi_j(K_j) = K_i \cap \varphi_i^{-1}\varphi_j(K_j). \end{aligned} \tag{2.6.1}$$

For each $j \in J$ the set K_{ij} is contained in $\varphi_i^{-1}\varphi_j(V_j)$. Thus each K_{ij} may be covered with open H_i-stable subsets Λ_{ij}^r of V_i such that there is an open embedding of orbifold charts $\lambda_{ij}^r \colon \Lambda_r \to V_j$. Since K_{ij} is compact, for each j there is a finite family $\{ \Lambda_{ij}^r | 1 \leq r \leq m_j \}$ which covers K_{ij}. As J_i is finite, we obtain for each $x \in K_i$ an open neighborhood

$$N_x := \bigcap_{x \in \Lambda_{ij}^r} \Lambda_{ij}^r \cap \left(V_i \setminus \bigcup_{j \in J, x \notin K_{ij}} K_{ij} \right).$$

Choose a H_i-stable connected open neighborhood $x \in Z^x \subseteq N_x$. Each $y \in Z^x$ is contained in K_{ij} only if x is contained in K_{ij} as well. For each $j \in J$, such that $x \in K_{ij}$ the open embeddings defined on Λ_{ij}^r restrict to an open embedding of orbifold charts on Z^x. Since K_i is compact we may select a finite open cover $\{ Z^{x_k} | x_k \in K_i,\ 1 \leq k \leq n \}$ of K_i. Observe that $Z^{x_k} \cap K_i \cap \varphi_i^{-1}\varphi_j(K_j) = Z^{x_k} \cap K_{ij}$ holds by (2.6.1). If this intersection is non-empty, we derive $x_k \in K_{ij}$. By construction there is an embedding of orbifold charts on Z^{x_k} which satisfies (b). Hence the family $\{ Z^{x_k} | x_k \in K_i,\ 1 \leq k \leq n \}$ satisfies all properties of assertion (b).
(c) follows directly from (b) and local compactness of each V_i: Before selecting a finite covering of the Z^x, we set $\hat{Z}^x := Z^x$ and choose for each x a compact neighborhood $x \in C_x \subseteq \hat{Z}^x$. The H_i-stable sets form a base of the topology and we may select a new H_i-stable subset $x \in Z_x \subseteq C_x^\circ \subseteq \hat{Z}^x$. By compactness of K_i we may select a finite covering from the family $(Z_x)_{x \in K_i}$ which satisfies (c). □

2.7. Examples of Orbifolds

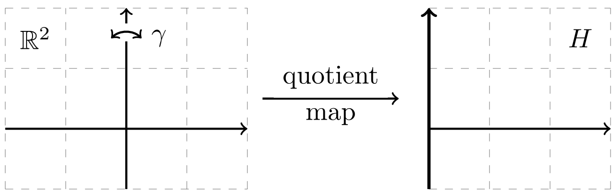
This section collects several well known examples from the literature to illustrate the definition of an orbifold. We introduce the following class of (trivial) examples to fix some terminology for later use:

2.7.1 Example Every paracompact smooth finite dimensional manifold $\mathcal{M}$ without boundary is an orbifold. An orbifold atlas for $\mathcal{M}$ is given by the following set of charts:

$$\{ (C, \{ \mathrm{id}_C \}, \mathrm{id}_C) | C \subseteq \mathcal{M} \text{ a connected component} \}$$

We call the orbifold structure induced on the manifold $\mathcal{M}$ the *trivial orbifold structure* .

2.7.2 Example (A mirror in $\mathbb{R}^2$ [53, 13.1.1, 13.2.2]) Consider $\mathbb{R}^2$ together with an action of the linear diffeomorphism $\gamma\colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (-x, y)$. The map γ fixes the points $(0, y), y \in \mathbb{R}$. An orbifold structure is induced on the quotient $\mathbb{R}^2/\langle\gamma\rangle \cong H := \{ (x, y) \in \mathbb{R}^2 \big| x \geq 0 \}$:



The boundary of the half plane contains the singular points, while points not on the boundary are regular points. This example can be generalized in the following way:
Let M be a (smooth) manifold with boundary ∂M glue together two copies of M along ∂M to obtain the double dM of M which is a manifold without boundary. Again the diffeomorphism which interchanges both halves of the double generates a finite group Γ. By construction the orbifold dM/Γ is isomorphic to M. Hence every manifold with boundary is in a natural way an orbifold, whose singular locus is the boundary of the manifold.

2.7.3 Example (Symmetric products [1, Example 1.13]) Suppose that M is a smooth manifold. Consider the *symmetric product* $X_n := M^n/S_n$, where M^n is the $n-$fold cartesian product of M and S_n the symmetric group of n letters which acts on M^n by permutation of coordinates. Tuples of points have non trivial isotropy groups if they contain a number of repetitions in their coordinates. The diagonal of M^n is fixed by each element of the finite group S_n.

2.7.4 Example (Orbifold structures on the 2-sphere) The following examples are all modelled on the 2-sphere $\mathbb{S}^2$, i.e. the topological space of each of the orbifolds is the 2-sphere with the topology turning it into a smooth manifold. Examples of this type first appeared in [53]. We give a detailed construction which follows [32]:
Let N be the north pole and S the south pole of $\mathbb{S}^2$. Endow the sphere with the usual topology turning $\mathbb{S}^2$ into a smooth manifold. Define charts around N respectively around S as follows:

Let $X_i := B^{\mathbb{R}^2}_{\frac{3}{4}\pi}(0)$, $i = 1, 2$ be the open disc of radius $\frac{3}{4}\pi$ centered at 0 in $\mathbb{R}^2$. We describe points in polar coordinates $(r, \theta), 0 \leq r < \frac{3}{4}\pi,\ 0 \leq \theta < 2\pi$. Recall that the geodesics connecting N and S on $\mathbb{S}^2$ are the great circles connecting N and S. To construct the charts pick a great circle C connecting N and S. Every great circle connecting N and S can uniquely be identified by an angle of rotation $0 \leq \theta < 2\pi$. Furthermore each x on $\mathbb{S}^2 \setminus \{ S \}$ is uniquely determined by a set of coordinates $(r, \theta),\ 0 \leq r < \pi, 0 \leq \theta < 2\pi$. Here r is the length of the geodesic segment between x and N. Analogously we may identify each point x in $\mathbb{S}^2 \setminus \{ N \}$ by a pair $(\pi - r, \theta), 0 \leq r < \pi, 0 \leq \theta < 2\pi$, where $\pi - r$ is the length of the geodesic segment between x and N. We obtain charts

$$\psi_1 \colon X_1 \to \mathbb{S}^2, (r, \theta) \mapsto (r, \theta), \quad \psi_2 \colon X_2 \to \mathbb{S}^2, (r, \theta) \mapsto (\pi - r, \theta)$$

for the manifold $\mathbb{S}^2$. These charts turn $\mathbb{S}^2$ into a smooth compact manifold in the usual way.
We construct an orbifold structure on $\mathbb{S}^2$: Let $n_i \in \mathbb{N}$ for $i = 1, 2$. Define the subgroup $G_i \subseteq \operatorname{Diff}(\mathbb{R}^2)$, which is generated by a rotation σ_i of order n_i, i.e. $\sigma_i.(r, \theta) := (r, \theta + \frac{2\pi}{n_i})$. Consider the quotient map $p_i \colon X_i \to X_i, (r, \theta) \mapsto (r, n_i\theta)$. A computation shows that this map is an n-fold covering of X_i, which identifies two points if and only if they are in the same G_i orbit. The maps ψ_i are diffeomorphisms, whence we obtain orbifold charts $(X_i, G_i, q_i),\ i = 1, 2$ with $q_i := \psi_i \circ p_i$. A computation shows that $A_{ij} := q_i^{-1}(\operatorname{Im} q_j) = \left\{ (r, \theta) \in X_i \middle| \frac{\pi}{4} \leq r \leq \frac{3\pi}{4} \right\}$ is an open annulus and furthermore the identities:

$$\tau_{ij} := q_i^{-1} \circ q_j \colon A_{ji} \to X_i,\ (r, \theta) \mapsto \left(\pi - r, \frac{n_j}{n_i} \cdot \theta\right),\ i \neq j \in \{ 1, 2 \} \tag{2.7.1}$$

The maps τ_{ij} are local diffeomorphisms which commutes with the orbifold charts, i.e. $q_i\tau_{ij} = q_j|_{\operatorname{dom} \tau_{ij}},\ i \neq j \in \{ 1, 2 \}$ holds. Locally the restrictions of the maps τ_{ij} thus yield change of chart morphisms. Since we obtain change of charts for each $x \in A_{ij}$, the orbifold charts are compatible and induce the structure

$$\mathcal{O}(n_1, n_2) := (\mathbb{S}^2, \{ (X_i, G_i, q_i) | i = 1, 2 \})$$

of a compact orbifold on $\mathbb{S}^2$. As a topological space, the base space of $\mathcal{O}(n_1, n_2)$ coincides with $\mathbb{S}^2$ with the usual topology. We distinguish the following cases:

$n_1, n_2 = 1$ In this case we have $q_i = \psi_i,\ i = 1, 2$ and thus $\mathcal{O}(1, 1)$ is just the C^∞-manifold $\mathbb{S}^2$.

$n_1 > 1, n_2 = 1$ We obtain a cone shaped singularity of order n_1 in N, while S is a regular point. The orbifold $\mathcal{O}(n_1, 1)$ is called $\mathbb{Z}_{n_1}$*-teardrop.*

$n_1 \neq n_2,\ n_1, n_2 > 1$ We obtain an orbifold with two cone shaped singularities of order n_1 respectively n_2. An orbifold of this kind is called $\mathbb{Z}_{n_1} - \mathbb{Z}_{n_2}$*-football.*

$n_1, n_2 = n > 1$ Consider an action of a finite group of diffeomorphisms $\Gamma \subset \operatorname{Diff}(\mathbb{S}^2)$ generated by a rotation of order n on $\mathbb{S}^2$. The group Γ acts smoothly, effectively and almost free on $\mathbb{S}^2$. Hence the orbit space $\mathbb{S}^2/\Gamma$ is an orbifold using the global orbifold chart $\pi \colon \mathbb{S}^2 \to \mathbb{S}^2/\Gamma$. By construction the orbifold structure of this orbifold agrees with $\mathcal{O}(n, n)$.

2.7.5 Example ([51, Example 2.4]) Let $Q := [0,1[$ be the topological space with the induced topology of $\mathbb{R}$. The map $f\colon Q \to Q, x \mapsto x^2$ is a homeomorphism. Consider the map $p\colon]-1,1[\to Q, x \mapsto |x|$. Let $\rho\colon \mathbb{R} \to \mathbb{R}$ be the reflection in 0. Then p induces a homeomorphism $]-1,1[/\langle\rho\rangle$. We derive orbifold charts $V_1 := (]-1,1[, \langle\rho\rangle, p)$ and $V_2 := (]-1,1[, \langle\rho\rangle, f \circ p)$.
We claim that these orbifold charts are not compatible. Assume to the contrary that they are compatible. Since $f \circ p(0) = 0 = p(0)$ there exist open connected neighborhoods U_1, U_2 of 0 in $]-1,1[$ and a diffeomorphism $h\colon U_1 \to U_2$ such that $f \circ p = p \circ h$. This equation leads to $h(x) \in \{\pm x^2\}$. By continuity we have the following choices for h:

$$h_1(x) := x^2 \qquad\qquad h_2(x) := -x^2$$

$$h_3(x) := \begin{cases} -x^2, & x \leq 0 \\ x^2, & x \geq 0 \end{cases} \qquad\qquad h_4(x) := \begin{cases} x^2, & x \leq 0 \\ -x^2, & x \geq 0 \end{cases}$$

Since none of the above is a diffeomorphism, the two charts are not compatible.

3. Maps of orbifolds

In this section we will discuss maps of orbifolds as defined in [51]. For the readers convenience, we repeat the definitions and constructions of [51] in Appendix E. Our goals in this section are as follows: We obtain a characterization of orbifold diffeomorphisms. Then several tools and constructions (such as open suborbifolds and orbifold partitions of unity) for later chapters are provided.

3.1. Orbifold Diffeomorphisms

In this section let $(Q_i, \mathcal{U}_i),\ i = 1, 2$ be arbitrary orbifolds. Obviously diffeomorphisms of orbifolds are the isomorphisms in the category of reduced orbifolds:

3.1.1 Definition A morphism of orbifolds $[\hat{f}] \in \mathbf{Orb}\,((Q_1, \mathcal{U}_1), (Q_2, \mathcal{U}_2))$ is called *orbifold diffeomorphism*, if there is $[\hat{g}] \in \mathbf{Orb}((Q_2, \mathcal{U}_2), (Q_1, \mathcal{U}_1))$, such that $\mathrm{id}_{(Q_1,\mathcal{U}_1)} = [\hat{g}] \circ [\hat{f}]$ and $\mathrm{id}_{(Q_2,\mathcal{U}_2)} = [\hat{f}] \circ [\hat{g}]$ hold. In this case, we also write $[\hat{f}]^{-1} := [\hat{g}]$.
By $\mathrm{Diff}_{\mathrm{Orb}}\,((Q_2, \mathcal{U}_2), (Q_1, \mathcal{U}_1))$ we denote the set of orbifold diffeomorphism in $\mathbf{Orb}((Q_2, \mathcal{U}_2), Q_1, \mathcal{U}_1)$. To shorten our notation $\mathrm{Diff}_{\mathrm{Orb}}\,((Q, \mathcal{U}), (Q, \mathcal{U}))$ will be abbreviated as $\mathrm{Diff}_{\mathrm{Orb}}\,(Q, \mathcal{U})$.

To construct a Lie group structure on the diffeomorphism group of an orbifold, we need an alternative characterization of orbifold diffeomorphisms.

3.1.2 Proposition *Let $[\hat{f}] \in \mathbf{Orb}((Q_1, \mathcal{U}_1), (Q_2, \mathcal{U}_2))$ be a diffeomorphism of orbifolds. Each representative $\hat{f} = (f, \{ f_i \}, [P_f, \nu_f])$ satisfies the following properties:*

(a) *the map f is a homeomorphism and*
(b) *every local lift f_i of $\hat{f}$ is a local diffeomorphism.*

Proof. We first notice that since $[\hat{f}] \circ [\hat{f}]^{-1}$ resp. $[\hat{f}]^{-1} \circ [\hat{f}]$ are the respective identity morphisms, the maps $f \colon Q_1 \to Q_2$ and $f^{-1} \colon Q_2 \to Q_1$ (where f^{-1} is the underlying continuous map of $[\hat{f}]^{-1}$) are homeomorphisms since composition yields the identity on Q_2 resp. Q_1. Hence the first assertion is true.
Two representatives of the class $[\hat{f}]$ are related via lifts of the identity. Lifts of such mappings are local diffeomorphisms, whence locally lifts of different representatives of $[\hat{f}]$ are related via diffeomorphisms to each other. Thus the definition of $[\hat{f}]$ shows that it suffices to prove the second assertion for any representative $\hat{f}$ of $[\hat{f}]$.
Choose a non-singular point $p \in Q_1$. The complement of the singular locus is open (cf. 2.5.3) and by Lemma 2.4.1 there is some chart $(V, G, \pi) \in \mathcal{U}_1$, such that $p \in \tilde{V} := \pi(V) \subseteq Q_1 \setminus \Sigma_{Q_1}$ and $G \cong \Gamma_p \cong \{ \mathrm{id}_V \}$. The map f is a homeomorphism, therefore $f(\tilde{V}) \subseteq Q_2$ is an open set and since the non-singular set is dense and open in Q, there is a $q \in f(\tilde{V})$ which is non-singular. We

may again choose an orbifold chart (W, H, ψ), such that $q \in \tilde{W} := \psi(W) \subseteq f(\tilde{V}) \cap Q_2 \setminus \Sigma_{Q_2}$. Without loss of generality we may assume $H \cong \Gamma_q \cong \{\, \mathrm{id}_W \,\}$. Now the group action on V and W is trivial and thus π and ψ are homeomorphisms. We have constructed an open embedding $\psi^{-1} f \pi \colon V \supseteq (f\pi)^{-1}(\psi(W)) \to W$. Composing with suitable charts of the manifolds V and W, this yields a homeomorphism of an open subset of $\mathbb{R}^{\dim V}$ onto an open subset of $\mathbb{R}^{\dim W}$. The invariance of domain theorem (cf. [34, 2B.3]) now forces $\dim V = \dim W$. In particular the dimensions of the orbifolds must coincide.
Choose and fix representatives $\mathcal{V} := \{\, (V_i, G_i, \pi_i) | i \in I \,\}$ of $\mathcal{U}_1$, $\mathcal{U} := \{\, (U_j, H_j, \psi_j) | j \in J \,\}$ of $\mathcal{U}_2$ and a range atlas $\mathcal{W} := \{\, (W_k, L_k, \varphi_k) | k \in K \,\} \subseteq \mathcal{U}_1$, such that the maps $[\hat{f}]$ and $[\hat{f}]^{-1}$ possess representatives $\hat{f} \in \mathrm{Orb}(\mathcal{V}, \mathcal{U})$, respectively $\hat{f}^{-1} \in \mathrm{Orb}(\mathcal{U}, \mathcal{W})$. Let $\alpha \colon I \to J$ and $\beta \colon J \to K$ be the unique maps, such that the lifts f_i resp. f_j^{-1} are local lifts w.r.t. the charts (V_i, G_i, π_i) and $(U_{\alpha(i)}, G_{\alpha(i)}, \psi_{\alpha(i)})$ resp. local lifts w.r.t. (U_j, G_j, ψ_j) and $(W_{\beta(j)}, G_{\beta(j)}, \varphi_{\beta_j})$. To shorten the notation set $\tilde{V}_i := \pi_i(V)$ and derive for every $i \in I$ a commutative diagramm:

$$\begin{tikzcd}
V_i \arrow[r, "f_i"] \arrow[d, "\pi_i"] & U_{\alpha(i)} \arrow[r, "f^{-1}_{\alpha(i)}"] \arrow[d, "\psi_{\alpha(i)}"] & W_{\beta(\alpha(i))} \arrow[d, "\varphi_{\beta\alpha(i)}"] \\
\tilde{V}_i \arrow[r, "f|_{\tilde{V}_i}"] & \tilde{U}_{\alpha(i)}) \arrow[r, "f^{-1}|_{\tilde{U}_{\alpha(i)}}"] & \tilde{W}_{\beta\alpha(i)}
\end{tikzcd}$$

Composition in the lower row induces the identity $\mathrm{id}_{Q_1}|_{\pi_i(V_i)}$. In particular for every $v \in V_i$ we have $\pi_i(v) = \varphi_{\beta\alpha(i)}(f^{-1}_{\alpha(i)} f_i(v))$. By compatibility of charts, there exists a restriction (V', H, χ) of (V_i, G_i, π_i), such that $v \in V'$ and there is an open embedding $\lambda \colon (V', H, \chi) \to (W_{\beta\alpha(i)}, G_{\beta\alpha(i)}, \varphi_{\beta\alpha(i)})$ with $\lambda(v) = f^{-1}_{\alpha(i)} f_i(v)$.
By continuity, there is an open H-stable neighborhood S of v with $S \subseteq (f^{-1}_{\alpha(i)} f_i)^{-1}(\lambda(V')) \cap V_i$. Let $g := \lambda^{-1} f^{-1}_{\alpha(i)} f_i|_S \colon S \to V'$ be the induced lift. By construction it is a local lift of id_{Q_1}. Using [48, Lemma 2.11] the map g is a diffeomorphism and the identity $\lambda \circ g = f^{-1}_{\alpha(i)} f_i|_S$ shows that $f_i|_S$ is injective with a continuous inverse given by $(\lambda \circ g)^{-1} f^{-1}_{\alpha(i)}$. In particular $f_i|_S$ is a diffeomorphism onto its image and by the invariance of domain theorem the image has to be open, since we have already shown that the dimensions of the orbifolds coincide. □

3.1.3 Corollary *Two orbifolds $(Q_i, \mathcal{U}_i)$, $i = 1, 2$ which are isomorphic have the same orbifold dimension. Thus every orbifold has a welldefined dimension and the orbifold dimension is an invariant preserved by orbifold diffeomorphisms.*

3.1.4 Definition Consider an orbifold map $[\hat{f}] \in \mathbf{Orb}((Q_1, \mathcal{U}_1), (Q_2, \mathcal{U}_2))$ together with a corresponding representative of orbifold maps $\hat{f} = (f, \{\, f_i \,\}, [P_f, \nu_f])$. We say that $[\hat{f}]$ *preserves local groups* if $f \colon Q_1 \to Q_2$ maps every element p of Q_1 onto some element $f(p)$ of Q_2 such that $\Gamma_p(Q_1) \cong \Gamma_{f(p)}(Q_2)$.
This property may be interpreted as preservation of the local structure of an orbifold. In particular one would expect that this is a natural property of orbifold diffeomorphisms. Indeed this is true, as the following proposition shows:

3.1.5 Proposition *Let $[\hat{f}]\colon (Q_1,\mathcal{U}_1) \to (Q_2,\mathcal{U}_2)$ be a map of orbifolds, with a representative $\hat{f} = (f, \{ f_i \}_{i\in I}, P_f, \nu_f)$, such that f is a homeomorphism and each f_i is a local diffeomorphism. Then $[\hat{f}]$ preserves local groups. In particular every orbifold diffeomorphism preserves local groups.*

Proof. Fix a representative $\hat{f}$ and some representative (P_f, ν_f) of $[P_f, \nu_f]$. Let p be in Q_1. There are orbifold charts $(V, G, \pi) \in \mathcal{U}_1$ and $(U, H, \psi) \in \mathcal{U}_2$ together with a local lift $f_V\colon V \to W$ of $\hat{f}$, such that $p \in \tilde{V}$, $q := f(p) \in \tilde{U}$. Fix some preimage $\hat{p} \in \pi^{-1}(p)$ and denote by $\hat{q} := f_V(\hat{p})$ its image. It suffices to show, that the isotropy group of $\hat{p}$ is isomorphic (as a group) to the isotropy group of $\hat{q}$. Then the same holds for the local groups of p resp. q.
Since $G_{\hat{p}}$ is finite, there is an open connected neighborhood Ω of $\hat{p}$ in V, such that for every $\gamma \in G_{\hat{p}}$, there is some $\mu_\gamma \in P_f$ with $\gamma|_\Omega = \mu_\gamma|_\Omega$. Thus one obtains

$$f_V(\gamma.x) = \nu_f(\mu_\gamma) f_V(x) \ \forall x \in \Omega, \gamma \in \Gamma_{\hat{p}}$$

Shrinking Ω if necessary, we may assume Ω to be a G-stable open connected subset with $G_\Omega = \Gamma_{\hat{p}}$ and $f_V|_\Omega$ is a diffeomorphism onto an open subset of U (the second property follows from Proposition 3.1.2). The map $\nu_f(\mu_\gamma)|_{f(\Omega)}\colon f(\Omega) \to U$ is a smooth map defined on an open connected subset of the manifold U. For $x \in \Omega$ we compute

$$\psi(\nu_f(\mu_\gamma) f_V(x)) = \psi f_V(\gamma.x) = f\pi(\gamma.x) = f\pi(x) = \psi f_V(x)$$

Thus $\nu_f(\mu_\gamma)|_{f(\Omega)}$ satisfies the identity $\psi(\nu_f(\mu_\gamma)(y)) = \psi(y),\ \forall y \in f(\Omega)$ and by [48, Lemma 2.11] there is a unique $h_\gamma \in H$ with $h_\gamma|_{f(\Omega)} = \nu_f(\mu_\gamma)|_{f(\Omega)}$. Furthermore since every $\gamma \in G_{\hat{p}}$ fixes $\hat{p}$, every h_γ has to fix $\hat{q}$. We derive a map

$$\theta\colon G_{\hat{p}} \to H_{\hat{q}}, \gamma \mapsto h_\gamma$$

Consider two elements $\gamma, \gamma' \in H$, such that $\operatorname{germ}_y \gamma = \operatorname{germ}_y \gamma'$ for some $y \in U$. As the germs are equal, there is an open (connected) subset on which γ and γ' coincide. Hence [48, Lemma 2.10 and Lemma 2.11] imply $\gamma = \gamma'$. Observe that every element of H is uniquely determined by its germ at an arbitrary point. Consider $\gamma, \eta, \rho \in G_{\hat{p}}$ such that $\gamma|_\Omega \eta|_\Omega = \rho|_\Omega$. Property (R4c) in definition E.2.3 yields $\operatorname{germ}_{f_V(\eta(x))} \nu_f(\mu_\gamma) \cdot \operatorname{germ}_{f_V(x)} \nu_f(\mu_\eta) = \operatorname{germ}_{f_V(x)} \nu_f(\mu_\rho),\ \forall x \in \Omega$ or in other words

$$\operatorname{germ}_{f_V(x)} h_\gamma \circ h_\eta = \operatorname{germ}_{f_V(\eta(x))} h_\gamma \cdot \operatorname{germ}_{f_V(x)} h_\eta = \operatorname{germ}_{f_V(x)} h_\rho,\ x \in \Omega$$

As the germs coincide, our previous observations imply $h_\gamma \cdot h_\eta = h_\rho$. Now $\theta(\gamma \cdot \eta) = \theta(\rho) = h_\rho = h_\gamma \cdot h_\eta = \theta(\gamma) \cdot \theta(\eta)$ and since $\gamma, \eta, \rho \in G_{\hat{p}}$ where chosen arbitrarily, θ is a group homomorphism. We claim that θ is injective. To see this choose $x \in \Omega \setminus \Sigma_G$ (which exists, since Ω is open) and observe that $\gamma.x = \eta.x$ if and only if $\gamma = \eta$ for all $\gamma, \eta \in G$. Since $f_V|_\Omega$ is a diffeomorphism onto its image and Ω is $G_{\hat{p}}$ invariant, we derive for $\gamma, \eta \in G_{\hat{p}}$:

$$h_\gamma . f_V(x) = f_V(\gamma.x) \neq f_V(\eta.x) = h_\eta . f_V(x)$$

and thus $h_\gamma \neq h_\eta$ if $\gamma \neq \eta$. This proves that θ is an injective group homomorphism. We claim that θ is surjective: Let $h \in H_{\hat{q}}$ be arbitrary. Since h fixes $\hat{q} = f_V(\hat{p})$, the set $h.f_V(\Omega) \cap f_V(\Omega)$ is open and non-empty. Newmans theorem B.2.1 assures that there is an element $y \in h.f_V(\Omega) \cap f_V(\Omega)$ whose isotropy subgroup H_y is trivial. We compute $\pi((f_V|_\Omega^{f_V(\Omega)})^{-1}(h^{-1}.y) = f^{-1}\psi(h^{-1}.y) = f^{-1}\psi(y) =$

$\pi(f_V|_\Omega^{f_V(\Omega)})^{-1}(y)$. The elements $(f_V|_\Omega)^{-1}(y)$ and $(f_V|_\Omega)^{-1}(h^{-1}.y)$ are in Ω and in the same G-orbit. By G-stability of Ω, there is a $\gamma \in G_{\hat{p}}$ with $\gamma(f_V|_\Omega)^{-1}(h^{-1}.y) = (f_V|_\Omega)^{-1}(y)$. Thus $h_\gamma h^{-1}.y = y$ holds and since H_y is trivial we derive $h_\gamma = h$. In conclusion θ is a group isomorphism, whence $G_{\hat{p}}$ and $H_{\hat{q}}$ are isomorphic. □

3.1.6 Remark The proof of proposition 3.1.5 does provide more informations about an orbifold map which satisfies the prerequisites of this proposition. Reviewing the proof, for the local lifts f_i w.r.t. (V_i, G_i, π_i) and (W_i, H_i, ψ_i) of such a map and $x \in V_i$, there is an arbitrarily small open subset Ω_x with the following properties:

(a) $f_i|_{\Omega_x}$ is a diffeomorphism onto an open set $\Omega_{f_i(x)} := f_i(\Omega_x)$,
(b) the set Ω_x is G_i-stable with $G_{i,\Omega_x} = G_{i,x}$,
(c) for each $\gamma \in G_{i,x}$ the restriction $\gamma|_{\Omega_x}$ is an element of P_f
(d) the set $\Omega_{f_i(x)}$ is H_i-stable with $H_{i,\Omega_{f_i(x)}} = H_{i,f_i(x)}$

In particular $(\Omega_x, G_{i,x}, \pi_i|_{\Omega_x})$ and $(\Omega_{f_i(x)}, H_{i,f_i(x)}, \psi_i|_{\Omega_{f_i(x)}})$ are orbifold charts contained in $\mathcal{U}_1$ resp. $\mathcal{U}_2$. Locally we may therefore always construct lifts which are diffeomorphisms.

It is possible to construct a charted orbifold map from a family of local lifts as given in the last remark:

3.1.7 Lemma *Let $f\colon Q_1 \to Q_2$ be a homeomorphism and $\{ f_i \}_{i\in I}$ a family of local lifts of f with respect to charts $\mathcal{V}$ and $\mathcal{W}$, such that each f_i is a local diffeomorphism. Assume that $\mathcal{V}$ satisfies (R2). Then there exists a pair (P, ν) such that $(f, \{ f_i \}_{i\in I}, P, \nu) \in \mathrm{Orb}(\mathcal{V}, \mathcal{W})$ is a representative of an orbifold map in* $\mathbf{Orb}((Q_1,\mathcal{U}_1),(Q_2,\mathcal{U}_2))$*. The pair (P,ν) is unique up to equivalence.*

Proof. Let $\mathcal{V} := \{ (V_i, G_i, \pi_i) | i \in I \}$ be the representative of $\mathcal{U}_1$, such that every f_i is a map $f_i\colon V_i \to W_i$ for some $(W_i, H_i, \psi_i) \in \mathcal{U}_2$. As f is a homeomorphism, $\mathcal{W} := \{ (W_i, H_i, \psi_i) | i \in I \}$ is an orbifold atlas. Define $F := \coprod_{i\in I} f_i$. Consider the set

$$P := \{ h \in \mathbf{\Psi}(\mathcal{V}) | h \text{ is a change of charts and } F|_{\operatorname{dom} h}, F|_{\operatorname{cod} h} \text{ are smooth embeddings} \}.$$

Clearly P is a quasi-pseudogroup which generates $\mathbf{\Psi}(\mathcal{V})$. Construct a map $\nu\colon P \to \mathbf{\Psi}(\mathcal{W})$ as follows: For $\lambda \in P$ there are $i, j \in I$, such that $\operatorname{dom}\lambda \subseteq V_i$ and $\operatorname{cod}\lambda \subseteq V_j$. $F|_{\operatorname{dom}\lambda} = f_i|_{\operatorname{dom}\lambda}$ is a diffeomorphism onto an open set $U_\lambda \subseteq W_i$. We may now define

$$\nu(\lambda) := f_j \lambda f_i|_{U_\lambda}^{-1}\colon U_\lambda \to f_j(\operatorname{cod}\lambda)$$

The set $f_j(\operatorname{cod}\lambda)$ is open since f_j is a local diffeomorphism. Following the definition of P, $\nu(\lambda)$ is a diffeomorphism. We compute $\psi_j \nu(\lambda) = \psi_j f_j \lambda f_i^{-1} = f \pi_j \lambda f_i^{-1} = f \pi_i f_i|_{U_\lambda}^{-1} = f f^{-1} \psi_i|_{U_\lambda} = \psi_i|_{U_\lambda}$, which shows $\nu(\lambda) \in \mathbf{\Psi}(\mathcal{W})$. In addition $F \circ \lambda = f_j \circ \lambda = \nu(\lambda) \circ f_i = \nu(\lambda) \circ F|_{\operatorname{dom}\lambda}$. Thus we have constructed a quasi-pseudogroup P and a well-defined map $\nu\colon P \to \mathbf{\Psi}(\mathcal{W})$ satisfying property (R4a) of definition E.2.3. Reviewing (R4b)-(R4d) of the same definition, clearly these properties

are satisfied by ν. In conclusion $(f, \{ f_i \}_{i\in I}, P, \nu)$ is a representative of an orbifold map.
To prove the uniqueness, assume that there is another pair (P', ν') turning $(f, \{ f_i \}_{i\in I}, (P', \nu'))$ into a charted map. Pick $\lambda \in P$, $\mu \in P'$ such that $x \in \operatorname{dom}\lambda \cap \operatorname{dom}\mu$ with $\operatorname{germ}_x \lambda = \operatorname{germ}_x \mu$. Assume that $x \in V_i$, $F(x) = f_i(x) \in W_i$ and $\mu(x) = \lambda(x) \in V_j$. Choose a stable subset $F(x) \in S \subseteq \operatorname{dom}\nu(\lambda) \cap \operatorname{dom}\nu'(\mu)$, such that $\nu(\lambda)|_S, \nu'(\mu)|_S$ are embeddings of orbifold charts. Then there is $\gamma \in H_j$ with $\gamma.\nu(\lambda)|_S = \nu'(\mu)|_S$. By Newmans Theorem B.2.1, we may pick a non singular $y \in f_i^{-1}(S) \cap \operatorname{dom}\mu \cap \operatorname{dom}\lambda$ such that $\mu(y) = \lambda(y)$ follows from $\operatorname{germ}_x \lambda = \operatorname{germ}_x \mu$. By definition we have $\gamma.\nu(\lambda)f_i(y) = \nu'(\mu)f_i(y) = f_j\mu(y) = f_j\lambda(y) = \nu(\lambda)f_i(y)$. Each f_i maps non-singular points to non singular points by Proposition 3.1.5 and thus $\nu(\lambda)f_i(y)$ is non singular, whereas $\gamma = \operatorname{id}_{W_j}$ holds. We conclude $\operatorname{germ}_{F(x)} \nu(\lambda) = \operatorname{germ}_{F(x)} \nu'(\mu)$ and thus $(P', \nu') \in [P, \nu]$ by definition E.2.5. □

Combining remark 3.1.6 with Lemma 3.1.7, we obtain the following corollary:

3.1.8 Corollary *Let $f\colon Q_1 \to Q_2$ be a homeomorphism and $\{ g_i \}_{i\in I}$ a family of local lifts of f with respect to atlases $\mathcal{V}'$ and $\mathcal{W}'$, such that each g_i is a local diffeomorphism. Assume that $\mathcal{V}'$ satisfies (R2). There exist orbifold atlases $\mathcal{V}$ which refines $\mathcal{V}'$ indexed by some J and an orbifold atlas $\mathcal{W}$ which refines $\mathcal{W}'$ and a family of lifts f_j w.r.t. $(V_j, G_j, \psi_j) \in \mathcal{V}'$, $(W_{\beta(j)}, H_{\beta(j)}, \varphi_{\beta(j)}) \in \mathcal{W}'$ such that each f_j is a diffeomorphism. In addition there is a unique equivalence class $[P, \nu]$ with $P = \mathcal{C}h_{\mathcal{V}'}$ and $\nu(\lambda) := f_k\lambda f_j|^{-1}_{f_j(\operatorname{dom}\lambda)}$ for $\lambda \in \mathcal{C}h_{V_j,V_k}$, $(V_r, G_r, \psi_r) \in \mathcal{V}$, $r = j,k$ such that $\hat{f} := (f, \{ f_j | j \in J \}, [P, \nu]) \in \operatorname{Orb}(\mathcal{V}, \mathcal{W})$.*

3.1.9 Lemma *Let $\mathcal{V} := \{ (V_i, G_i, \psi_i) | i \in I \}$ and $\mathcal{W} := \{ (W_j, H_j, \varphi_j) | j \in J \}$ be atlases for orbifolds $(Q_1, \mathcal{U}_1)$ resp. $(Q_2, \mathcal{U}_2)$. Consider a charted map of orbifolds $\hat{f} := (f, \{ f_i | i \in I \}, [P, \nu]) \in$* $\operatorname{Orb}(\mathcal{V}, \mathcal{W})$ *with the same properties as the map $\hat{f}$ in corollary 3.1.8. Then the following holds:*

(a) *For each G_i-stable subset $\Omega \subseteq V_i$ the set $f_i(\Omega)$ is a $H_{\beta(i)}$-stable subset of $W_{\beta(i)}$ with isotropy group $H_{\beta(i), f_i(\Omega)} \cong G_{i,\Omega}$*
(b) *After possibly shrinking $\mathcal{V}$ and $\mathcal{W}$, we may assume that the map[2] $\beta\colon I \to J$ is bijective,*
(c) *If β is bijective then $\nu\colon \mathcal{C}h_{\mathcal{V}} \to \mathcal{C}h_{\mathcal{W}}$ is a bijection,*

Proof. (a) Let $\Omega \subseteq V_i$ be a G_i-stable subset with isotropy subgroup $G_{i,\Omega}$. Then each $\gamma \in G_i$ induces a change of charts morphism $\nu(\gamma) = f_i\gamma f_i^{-1}\colon W_{\beta(i)} \to W_{\beta(i)}$. By [48, Lemma 2.11] there is $h_\gamma \in H_{\beta(i)}$ with $h_\gamma = \nu(\gamma)$. Vice versa, an analogous argument may be given to associate to each $h \in H_{\beta(i)}$ a group element $g_h := f_i^{-1}hf_i \in G_i$. By G_i-stability of Ω we observe that $h_\gamma \in H_{\beta(i), f_i(\Omega)}$ holds for each $\gamma \in G_{i,\Omega}$ and obtain a map $\theta\colon G_{i,\Omega} \to H_{\beta(i),\Omega}, \gamma \mapsto h_\gamma$. As $\theta(\gamma) \circ \theta(\eta) = h_\gamma \circ h_\eta|_{f_i(\Omega)} = f_i\gamma\eta f_i^{-1} = h_{\gamma\circ\eta} = \theta(\gamma \circ \eta)$ we conclude that θ is a group homomorphism. The map f_i is a diffeomorphism which implies that $\theta(\gamma) = \theta(\eta)$ forces $\gamma = \eta$ and therefore θ is injective. We are left to prove that $f_i(\Omega)$ is $H_{\beta(i)}$-stable and θ is surjective. To this end consider $h \in H_{\beta(i)}$ such that $h.f_i(\Omega) \cap f_i(\Omega)$ is non-empty. Choose $x, y \in \Omega$ with $h.f_i(y) = f_i(x) \in h.f_i(\Omega) \cap f_i(\Omega)$ and compute $g_h.y = f_i^{-1}(h.f_i(y)) = x \in \Omega$. Now by

[2] which assignes to each index i an index $\beta(i) \in J$ such that $g_i\colon V_i \to W_{\beta(i)}$ holds.

G_i-stability of Ω, $g_h \in G_{i,\Omega}$ and therefore $h = \theta(g_h) \in H_{\beta(i),f_i(\Omega)}$. We conclude that $f_i(\Omega)$ is $H_{\beta(i)}$-stable and its isotropy group is isomorphic to $G_{i,\Omega}$.

(b) Assume there are $i, j \in I$ with $\beta(i) = \beta(j)$, we obtain a diffeomorphism $f_j^{-1} f_i \colon V_i \to V_j$. A quick computation shows that $\psi_j f_j^{-1} f_i = f^{-1}\varphi_{\beta(j)} f_i = \psi_i$ and thus $f_j^{-1} f_i$ is an open embedding of orbifold charts. Reversing the roles of i and j, also $f_i^{-1} f_j$ is an embedding of orbifold charts. Therefore we may ommit one index of the pair i, j with $\beta(i) = \beta(j)$ and the set of orbifold charts indexed by the reduced set will again be an orbifold atlas. The axiom of choice allows us to shrink $\mathcal{V}$ to obtain an orbifold atlas (which by abuse of notation will also be called $\mathcal{V}$) such that β is injective. Clearly since f is a homeomorphism, the set of charts $\{ (W_j, H_j, \varphi_j) \in \mathcal{W} | j = \beta(i) \text{ for some } i \in I \}$ is an orbifold atlas. Thus by replacing J with $\beta(I)$, we may assume that β is surjective, hence bijective.

(c) It is obvious that ν is injective. Let $\lambda \in \mathcal{C}h_{W_k,W_l}$ be any change of charts morphism with $(W_r, H_r, \varphi_r) \in \mathcal{W}$, $r = k, l$. There are unique $i, j \in I$ with $\beta(i) = k$ and $\beta(j) = l$ and we optain a diffeomorphism $\mu(\lambda) := f_j^{-1} \lambda f_i|_{f_i^{-1}(\operatorname{dom}\lambda)} \colon f_i^{-1}(\operatorname{dom}\lambda) \to f_j^{-1}(\operatorname{cod}\lambda)$. A quick computation leads to $\psi_j \mu(\lambda) = f^{-1}\varphi_l \lambda f_i|_{f_i^{-1}(\operatorname{dom}\lambda)} = f^{-1} f \psi_i|_{\operatorname{dom}\lambda} = \psi_i|_{\operatorname{dom}\lambda}$ which proves that $\mu(\lambda) \in \mathcal{C}h_{V_i,V_j}$. By construction $\nu(\mu(\lambda)) = \lambda$ holds and thus ν is a bijection.

□

The next proposition is the converse of Proposition 3.1.2, i.e. we shall prove that the characterization of orbifold diffeomorphisms in Proposition 3.1.2 is equivalent to the categorical definition. The leading idea is to use the local properties of the lifts (i.e. that every lift may locally be inverted) to construct a family of lifts for f^{-1}. In general a given lift may not be inverted globally. Nevertheless it is possible to construct smaller charts and induced lifts, which may be inverted globally.

3.1.10 Proposition *Let $\hat{f} := (f, \{ f_i \}, [P, \nu]) \in \operatorname{Orb}(\mathcal{V}, \mathcal{W})$ be an orbifold map. If f is a homeomorphism and $f_i \colon V_i \to W_{\alpha(i)}$ is a local diffeomorphism for each $i \in I$, the orbifold map $[\hat{f}] \in \mathbf{Orb}((Q_1, \mathcal{U}_1), (Q_2, \mathcal{U}_2))$ is a diffeomorphism of orbifolds.*

Proof. Combining Corollary 3.1.8 and Lemma 3.1.9 there are orbifold atlases $\mathcal{V}'$ indexed by J and $\mathcal{W}'$ indexed by K together with a representative $\hat{g} := (f, \{ g_j \} j \in J, [P, \nu]) \in \operatorname{Orb}(\mathcal{V}', \mathcal{W}')$ of $[\hat{f}]$, such that each lift $g_j \colon V_j \to W_{\beta(j)}$ is a diffeomorphism and the map $\beta \colon J \to K$ is a bijection. We use the computation from the proof of Lemma 3.1.9: The inverse $g_j^{-1} \colon W_{\beta(j)} \to W_j$ of g_j is a local lift of f^{-1} w.r.t. $(W_{\beta(j)}, H_{\beta(j)}, \varphi_{\beta(j)})$ and (V_j, G_j, ψ_j). Since f is a homeomorphism, the family $\mathcal{W}'$ is an atlas for Q_2 indexed by K. As each g_j^{-1} is a diffeomorphism, by Lemma 3.1.7 there is a pair $P \subseteq \mathbf{\Psi}(\mathcal{W}')$ and $\nu \colon P \to \mathbf{\Psi}(\mathcal{V}')$ such that $\hat{h} := (f^{-1}, \{ g_j^{-1} \}_{j \in K}, P, \nu) \in \operatorname{Orb}(\mathcal{W}, \mathcal{V})$.

Consider the compositions $\hat{h} \circ \hat{g}$ and $\hat{g} \circ \hat{h}$: The local lift for every $j \in J$ of $\hat{g}$ has been constructed as inverse maps of the local lift of $\hat{g}$ w.r.t. (V_j, G_j, ψ_j) and $(W_{\beta(j)}, H_{\beta(j)}, \varphi_{\beta(j)})$. Thus the composition of both representatives gives a lift of the identity and we derive

$$[\hat{f}] \circ [\hat{g}] = [\hat{h} \circ \hat{g}] = \operatorname{id}_{(Q_2, \mathcal{U}_2)} \qquad [\hat{g}] \circ [\hat{f}] = [\hat{g} \circ \hat{h}] = \operatorname{id}_{(Q_1, \mathcal{U}_1)}$$

□

Observe that the proof of the last proposition yields the following fact: Assume that each member of the family of local lifts for an orbifold map is a diffeomorphism. Then this family uniquely determines the orbifold map. In particular each orbifold diffeomorphism is uniquely determined by its family of local lifts:

3.1.11 Corollary *An orbifold diffeomorphism* $[\hat{f}] \in \mathbf{Orb}((Q_1,\mathcal{U}_1),(Q_2,\mathcal{U}_2))$ *is uniquely determined by the family of local lifts* $\{ f_i | i \in I \}$ *where* $(f, \{ f_i \}_{i\in I}, P, \nu) \in [\hat{f}]$ *is an arbitrary representative.*

Summarizing the results in this section one obtains:

3.1.12 Corollary *For an orbifold map* $[\hat{f}] \in \mathbf{Orb}((Q_1,\mathcal{U}_1),(Q_2,\mathcal{U}_2))$ *the following are equivalent:*

(a) $[\hat{f}]$ *is an orbifold diffeomorphism,*
(b) *each representative* $(f, \{ f_i \}_{i\in I}, P, \nu) \in [\hat{f}]$ *satisfies:* f *is a homeomorphism and each* f_i *is a local diffeomorphism,*
(c) *there is a representative* $\hat{f} = (f, \{ f_i \}_{i\in I}, P, \nu)$ *of* $[\hat{f}]$*, such that* f *is a homeomorphism and each* f_i *is a local diffeomorphism*
(d) *there is a representative* $\hat{f} = (f, \{ f_j \}_{j\in J}, P, \nu) \in \mathrm{Orb}(\mathcal{V},\mathcal{W})$ *of* $[\hat{f}]$*, such that* f *is a homeomorphism and each* f_j *is a diffeomorphism. Furthermore the assignment* $\alpha\colon \mathcal{V} \to \mathcal{W}$ *such that* f_i *is a local lift w.r.t.* (V_i, G_i, φ_i) *and* $(W_{\alpha(i)}, G_{\alpha(i)}, \psi_{\alpha(i)})$ *is bijective.*

Let $\hat{f}$ *be as in (d), then a representative of* $[\hat{f}]^{-1}$ *is given by* $(f^{-1}, \{ f_j^{-1} \}, [\nu(P), \theta]) \in \mathrm{Orb}(\mathcal{W},\mathcal{V})$*. Here* $\theta\colon \nu(P) \to \mathbf{\Psi}(V)$ *assigns to* $\lambda \in \nu(P)$ *with* $\operatorname{dom}\lambda \subseteq W_{\alpha(i)}$ *and* $\operatorname{cod}\lambda \subseteq W_{\alpha(j)}$ *the map* $\theta(\lambda) := f_j^{-1}\lambda f_i|_{f_i^{-1}(\operatorname{dom}\lambda)}$.

3.2. Open Suborbifolds and restrictions of orbifold maps

As an application we define the notion of an open suborbifold. Any subset of a metrizable space with the induced topology is again a metrizable space. Every metrizable space is paracompact and Hausdorff by [20, Theorem 5.1.3]. Since the base space Q of the orbifold $(Q,\mathcal{U})$ is metrizable by Proposition 2.4.3, each of the subspaces in the following constructions will be a paracompact Hausdorff space.

3.2.1 Definition (open suborbifold) Let $(Q,\mathcal{U})$ be an orbifold. An orbifold $(X,\mathcal{X})$ is called *open suborbifold* of $(Q,\mathcal{U})$ if there is a map $[\hat{\iota}] = [(\iota, \{ \iota_k \}_{k\in I}, [P,\nu])] \in \mathbf{Orb}((X,\mathcal{X}),(Q,\mathcal{U}))$, such that

(a) ι is a topological embedding with open image,
(b) every ι_k is a local diffeomorphism.

A map ι with the properties (a) and (b) is called *open embedding of orbifolds.*

3.2.2 Definition (Restriction of an orbifold map to an open subset) Let $(Q,\mathcal{U})$ be an orbifold and $\Omega \subseteq Q$ be an open subset. The subset $\mathcal{U}_\Omega := \{ (V,G,\psi) \in \mathcal{U} | \psi(V) \subseteq \Omega \}$ is an orbifold atlas of Ω, turning the inclusion $\iota_\Omega \colon \Omega \hookrightarrow Q$ into an open embedding of orbifold. Define the restriction $[\hat{f}]|_\Omega$ of $[\hat{f}] \in \mathbf{Orb}((Q_1,\mathcal{U}_1),(Q_2,\mathcal{U}_2))$ to Ω via

$$[\hat{f}]|_\Omega := [\hat{f}] \circ [\iota_\Omega]$$

3.2.3 Definition (Corestriction of an orbifold map) Let $(X,\mathcal{X})$ be an open suborbifold of $(Q,\mathcal{U})$ together with open embedding of orbifolds $[\hat{\iota}]$. Consider another orbifold $(Q',\mathcal{V})$ and a map $[\hat{f}] \in \mathbf{Orb}((Q',\mathcal{V}),(Q,\mathcal{U}))$ with representative $\hat{f} = (f, \{ f_k \}_{k\in I}, (P,\nu) \in [\hat{f}]$, such that $\operatorname{Im} f \subseteq \operatorname{Im} \iota$.
For $k \in I$ let the lifts be given as $f_k \colon V_k \to U_{\alpha(k)}$ where $(U_{\alpha(k)}, G_{\alpha(k)}, \psi_{\alpha(k)})$ is an orbifold chart. Then $\operatorname{Im} f_k \subseteq \psi^{-1}_{\alpha(k)}(\operatorname{Im}\iota)$ holds. As $\operatorname{Im} f_k$ is connected it is contained in a connected component of the invariant set $\psi^{-1}_{\alpha(k)}(\operatorname{Im}\iota)$. The connected components of an invariant set are $G_{\alpha(k)}$-stable subsets of $U_{\alpha(k)}$. Hence these connected components can be made into orbifold charts for the subset $\operatorname{Im}\iota$. Using these charts, Lemma E.4.2 shows that there is a representative $\hat{g} \in \operatorname{Orb}(\mathcal{V}',\mathcal{U}')$ of $[\hat{f}]$, such that each lift $g_k \colon V'_k \to U'_k$ of $\hat{g}$ satisfies $\varphi(U'_k) \subseteq \operatorname{Im}\iota$. Define the *corestriction* of $[\hat{f}]$:

$$[\hat{f}]|^{\operatorname{Im}\iota} := \left[(f|^{\operatorname{Im}\iota}, \{ g_k \}_k, [P,\nu])\right] \in \mathbf{Orb}((Q',\mathcal{V}),(\operatorname{Im}\iota,\mathcal{U}|_{\operatorname{Im}\iota}))$$

In particular we obtain a unique map $([\hat{\iota}]|^{\operatorname{Im}\iota})^{-1} \circ [\hat{f}]|^{\operatorname{Im}\iota} \in \mathbf{Orb}((Q',\mathcal{V}),(X,\mathcal{X}))$ into the open suborbifold. By definition of the equivalence relation E.4.3, the class $[\hat{f}]|^{\operatorname{Im}\iota}$ does not depend on any choices made in the construction.

3.2.4 Remark

(a) An orbifold $(X,\mathcal{X})$ is an open suborbifold of $(Q,\mathcal{U})$ if and only if there is an orbifold diffeomorphism from $(X,\mathcal{X})$ to an orbifold which arises as the restriction of $\mathcal{U}$ to an open subset.
(b) Consider an open subset $\Omega \subseteq Q$ and the representative $\hat{f} = (f, \{ f_k \}_{k\in I}, [P,\nu])$ of $[\hat{f}] \in \mathbf{Orb}((Q,\mathcal{U}),(Q',\mathcal{W}'))$, such that there is $J \subseteq I$ with the following properties:
$\mathcal{V}_\Omega := \{ (V_j, G_j, \pi_j) \}_{j\in J} \subseteq \mathcal{U}_\Omega$ and $\Omega = \bigcup_{j\in J} \pi_j(V_j)$ hold.
Define $P_J := P \cap \mathcal{C}h_{\mathcal{V}_\Omega}$ and set $\nu_J := \nu|_{P_J}$. The composition in **Orb** is induced by composition of suitable representatives of these maps. Hence a computation with the representative above yields $[\hat{f}]|_\Omega = [\hat{h}]$, where $\hat{h} := (f|_\Omega, \{ f_j \}_{j\in J}, [P_J, \nu_J])$.
(c) Let $(X,\mathcal{X})$ be an open suborbifold with open embedding of orbifolds $[\hat{\iota}]$. By construction $[\hat{f}]|_{\operatorname{Im}\hat{\iota}} = [\hat{f}] \circ [\hat{\iota}] \circ ([\hat{\iota}]|^{\operatorname{Im}\iota})^{-1}$ holds.
(d) In section 4 tangent spaces of orbifolds and the tangent orbifold are defined. As those objects are defined via an arbitrary orbifold chart, analogous to the manifold case, for each open suborbifold $(X,\mathcal{X})$ of $(Q,\mathcal{U})$ the tangent spaces $T^{\mathcal{X}}_p X$ and $T^{\mathcal{U}}_{\iota(p)} Q$ are canonically isomorphic. If the open suborbifold is an open subset we shall identify the tangent spaces later on.

3.3. Partitions of unity for orbifolds

3.3.1 Definition Let $(Q,\mathcal{U})$ be an orbifold, $\mathcal{V} := \{ (V_i, G_i, \pi_i) | i \in I \}$ a representative of $\mathcal{U}$ and endow $\mathbb{R}$ with the trivial orbifold structure (i.e. the one induced by its manifold structure).
A family $\left\{ (\chi_i, \left\{ \chi_i^j \right\}_{j\in J}, [P_i, \nu_i]) \right\}_{i \in I} \subseteq \mathrm{Orb}(\mathcal{V}, \{ \mathrm{id}_{\mathbb{R}} \})$ is called *smooth orbifold partition of unity* subordinate to $\mathcal{V}$ if the set of continuous maps $\{ \chi_i \}_{i\in I}$ is a partition of unity subordinate to the open covering $\{ \pi_i(V_i) \}_{i \in I}$, i.e.

(a) $\operatorname{supp} \chi_i \subseteq V_i$ for all $i \in I$,
(b) the family $\{ \operatorname{supp} \chi_i \}_{i\in I}$ is locally finite,
(c) $\chi_i \geq 0,\ \forall i \in I$ and $\sum_{i\in I} \chi_i(x) = 1$ for each $x \in Q$

3.3.2 Proposition (Partition of Unity) *Let $(Q,\mathcal{U})$ be an orbifold then there exists a smooth partition of unity on $\mathcal{U}$. In particular, for each representative $\mathcal{V}$ of $\mathcal{U}$ there is a smooth orbifold partition of unity subordinate to $\mathcal{V}$.*

Proof. Each representative of $\mathcal{U}$ allows a locally finite refinement by Lemma 2.6.5 (b), thus the assertion will be true if the existence of a smooth orbifold partition of unity for an arbitrary locally finite representative of $\mathcal{U}$ can be verified.
Let $\mathcal{V} := \{ (U_\alpha, G_\alpha, \pi_\alpha) | \alpha \in I \}$ be a locally finite representative and $\tilde{\mathcal{V}} := \{ \pi_i(U_i) \}_{i\in I}$ be the family of open images of the charts in $\mathcal{V}$. Since Q is a paracompact Hausdorff space, applying [20, Lemma 5.1.6] twice, there is a locally finite family of open sets $\tilde{W}_\alpha^1 \subsetneq \overline{\tilde{W}_\alpha^1} \subsetneq \tilde{W}_\alpha^2 \subsetneq \overline{\tilde{W}_\alpha^2} \subsetneq \pi_\alpha(U_\alpha)$ such that $\left\{ \tilde{W}_\alpha^1 \middle| \alpha \in I \right\}$ covers Q (here the closure means closure in Q). Let $W_\alpha^i := \pi_\alpha^{-1}(\tilde{W}_\alpha^i),\ i = 1, 2.$
Observe that since $\overline{W}_\alpha^i \subsetneq \operatorname{Im} \pi_\alpha$, it is closed in the subspace topology. On $\operatorname{Im} \pi_\alpha^i$, we identify π_α with the quotient map onto the orbit space of the G_α-action on U_α. This map is surjective continuous, open and closed by Lemma B.1.4. Hence for $i = 1, 2$ [19, III. Theorem 8.3 (5) and Theorem. 11.4] imply $\pi_\alpha(\overline{W_\alpha^i}) = \overline{\tilde{W}_\alpha^i}$ and $\overline{W_\alpha^i} \subseteq \pi_\alpha^{-1}(\overline{\tilde{W}_\alpha^i})$. Vice versa [19, III. Theorem 11.2 (2)] yields $\overline{W_\alpha^i} = \pi_\alpha^{-1}\left(\overline{\tilde{W}_\alpha^i}\right)$. By construction, every set W_α^i is G_α-invariant by construction.
The manifold U_α is a smooth connected paracompact (hence second countable by Proposition 2.4.2) and finite dimensional manifold. By the smooth Urysohn Lemma (cf. [15, Corollary 3.5.5.]) for manifolds, there is a smooth map $f^\alpha \colon U_\alpha \to [0, 1]$ such that $f^\alpha|_{\overline{W_\alpha^1}} \equiv 1$ and $\operatorname{supp} f^\alpha \subsetneq W_\alpha^2$. Define an equivariant smooth map with values in $[0, 1]$ by averaging over G_α:

$$\theta_\alpha(y) := \frac{1}{|G_\alpha|} \sum_{\gamma\in G_\alpha} f^\alpha(\gamma.y).$$

Notice that $W_\alpha^1 \subset \operatorname{supp} \theta_\alpha \subseteq W_\alpha^2$ still holds by G_α-invariance of these sets. In particular the map vanishes outside of $\overline{W}_\alpha^2$. Define a map for every $\beta \in I$ as follows:

$$\theta_{\alpha,\beta} \colon U_\beta \to [0, 1],\ x \mapsto \begin{cases} \theta_\alpha(\pi_\alpha^{-1}\pi_\beta(x)) & \pi_\alpha^{-1}\pi_\beta(x) \neq \emptyset \\ 0 & \pi_\alpha^{-1}\pi_\beta(x) = \emptyset \end{cases}$$

By construction the G_α-equivariance of θ_α implies that $\theta_{\alpha,\beta}$ is a well-defined G_β-equivariant map. We claim that $\theta_{\alpha,\beta}$ is smooth: To see this note that for each $x \in \pi_\beta^{-1}(\operatorname{Im}\pi_\alpha)$ there is an open neighborhood $V_x \subseteq U_\beta$ of x and a smooth change of charts morphism $\lambda\colon V_x \to U_\alpha$. On the open set V_x, the map $\theta_{\alpha,\beta}$ is a composition of smooth maps: $\theta_{\alpha,\beta}|_{V_x} = \theta_\alpha \circ \lambda$. Hence on $\pi_\beta^{-1}(\operatorname{Im}\pi_\alpha)$ the map $\theta_{\alpha,\beta}$ is smooth.

By construction $\operatorname{supp}\theta_\alpha \subseteq \overline{W_\alpha^2} \subsetneq U_\alpha$ holds, i.e. we obtain $\pi_\beta(\operatorname{supp}\theta_{\alpha,\beta}) \subseteq \overline{\tilde{W}_\alpha^2} \subseteq \operatorname{Im}\pi_\alpha$. The above shows that $\theta_{\alpha,\beta}$ is a smooth map on the open neighborhood $\pi_\beta^{-1}(\operatorname{Im}\pi_\alpha)$ of its support. Outside of this set the map vanishes and in conclusion $\theta_{\alpha,\beta}$ is smooth.

Notice that by construction $\theta_{\alpha,\alpha} = \theta_\alpha$ holds. Since the family $\tilde{\mathcal{V}}$ is locally finite, for $x \in Q$ there are only finitely many $\alpha \in I$, such that $\pi_\alpha^{-1}(x) \neq \emptyset$. Define another G_β-equivariant smooth map on U_β:

$$\chi_{\alpha,\beta}\colon U_\beta \to [0,1], \chi_{\alpha,\beta} := \frac{\theta_{\alpha,\beta}}{\sum_{\delta\in I}\theta_{\delta,\beta}}$$

The map $\chi_{\alpha,\alpha}$ satisfies $\chi_{\alpha,\alpha}|_{U_\alpha\setminus\overline{W_\alpha^2}} \equiv 0$. Since every chart map is an open map, the map $\chi_{\alpha,\alpha}$ descends to a continuous map on Q:

$$\chi_\alpha\colon Q \to [0,1], x \mapsto \begin{cases} \chi_{\alpha,\alpha}\pi_\alpha^{-1}(x) & x \in U_\alpha \\ 0 & x \in Q \setminus U_\alpha \end{cases}$$

By construction $\operatorname{supp}\chi_\alpha \subseteq \pi_\alpha(U_\alpha)$. For every $\sigma \in I$ the smooth map $\chi_{\alpha,\sigma}$ is a lift of χ_α in the chart $(U_\sigma, G_\sigma, \pi_\sigma) \in \mathcal{V}$. The family $\tilde{\mathcal{V}}$ covers Q and we have constructed a family of continuous maps with smooth lifts in every orbifold chart of $\mathcal{V}$. As $[0,1]$ is a trivial orbifold, the following data completes the construction of an orbifold map: Choose the quasi-pseudogroup $P := \mathcal{C}h_{\mathcal{V}}$ which generates $\mathbf{\Psi}(\mathcal{V})$ and take $\nu\colon \mathcal{C}h_{\mathcal{V}} \to \mathbf{\Psi}(\mathbb{R}), f \mapsto \operatorname{id}_\mathbb{R}$. These choices lead to a map $(\chi_\alpha, \{\chi_{\alpha,\sigma}\}, [P,\nu])$ which clearly satisfies the requirements of definition E.2.5 (cf. E.2.6) and $\{\hat{\chi}_\alpha := (\chi_\alpha, \{\chi_{\alpha,\sigma}\}, [P,\nu])\}_{\alpha\in I} \subseteq \operatorname{Orb}(\mathcal{V}, \{\operatorname{id}_\mathbb{R}\})$ is a family of charted orbifold maps.

The construction of χ_α shows $\tilde{W}_1^\alpha \subseteq \operatorname{supp}\chi_\alpha \subseteq \pi_\alpha(U_\alpha)$ and the sets $\tilde{W}_\alpha^1$ cover Q. Thus the family $\{\operatorname{supp}\chi_\alpha\}_{\alpha\in I}$ covers Q and since $\tilde{\mathcal{V}}$ is locally finite, this family is locally finite. A quick computation now shows for $x \in Q$:

$$\begin{aligned}\sum_{\alpha\in I}\chi_\alpha(x) &= \sum_{\alpha\in I, x\in\pi_\alpha(U_\alpha)} \chi_{\alpha,\alpha}\pi_\alpha^{-1}(x) = \sum_{\alpha\in I, x\in\pi_\alpha(U_\alpha)} \frac{\theta_{\alpha,\alpha}}{\sum_{\delta\in I}\theta_{\delta,\alpha}} \circ \pi_\alpha^{-1}(x) \\ &= \sum_{\alpha\in I, x\in\pi_\alpha(U_\alpha)} \frac{\theta_\alpha\pi_\alpha^{-1}(x)}{\sum_{\delta\in I, x\in\pi_\delta(U_\delta)}\theta_\delta\pi_\delta^{-1}(x)} = 1\end{aligned}$$

The family $\{\chi_\alpha\}_{\alpha\in I}$ therefore is a partition of unity subordinate to $\mathcal{V}$. In conclusion, the family $\{\hat{\chi}_\alpha\}_{\alpha\in I}$ is a smooth orbifold partition of unity subordinate to $\mathcal{V}$. □

3.3.3 Notation Let $(Q,\mathcal{U})$ be an orbifold and $\{\hat{\chi}_\alpha\}$ an orbifold partition of unity subordinate to the locally finite orbifold atlas $\mathcal{V}$ indexed by I as in Proposition 3.3.2. For any pair $(\alpha,\beta) \in I \times I$, the lift of χ_α on U_β will be abbreviated as $\chi_{\alpha,\beta}$.

4. Tangent Orbibundles and their Sections

In this section we construct an analogon to tangent manifolds and tangent maps for an orbifold. Tangent orbifolds are well known objects (cf. [1, Proposition 1.21]). However the bundle map associated to a tangent orbifold turns out to be a map of orbifolds. This allows us to define orbisections, i.e. maps of orbifolds which are sections of the bundle map. In chapter 6.1 these orbisections will be the model space for the diffeomorphism group of an orbifold. Furthermore it is possible to construct a tangent endofunctor for the category of reduced (smooth) orbifolds. For the rest of this section let $(Q,\mathcal{U})$ be an orbifold. We begin with the construction of tangent orbifolds:

4.1. The Tangent Orbifold and the Tangent Endofunctor

4.1.1 Construction (Tangent space of an orbifold) Let $p \in Q$ and $(V_i, G_i, \pi_i) \in \mathcal{U},\ i = 1,2$ arbitrary orbifold charts with $p \in \pi_i(V_i)$. Consider pairs $(\pi_i, v_i),\ i = 1,2$ where $v_i \in T_{x_i}V_i$ with $x_i \in \pi_i^{-1}(p)$. Notice that by compatibility of orbifold charts, there exist open neighborhoods $x_i \in U_i \subseteq V_i$ and a change of charts morphism $\lambda\colon U_1 \to U_2$, such that $\lambda(x_1) = x_2$. Identify the tangent spaces $T_{x_i}V_i$ with the corresponding tangent spaces of the open submanifolds $U_i \subseteq V_i$. Since every change of charts morphism is a diffeomorphism the tangent spaces $T_{x_1}V_1$ and $T_{x_2}V_2$ are isomorphic. Introduce an equivalence relation on the set of all possible pairs of this kind: We declare two pairs to be equivalent $(\pi_1, v_1) \sim (\pi_2, v_2)$, if there are open subsets $x_i \in U_i \subseteq V_i$ and a change of charts morphism $\lambda\colon U_1 \to U_2$ such that $T\lambda(v_1) = v_2$. Here $T\lambda\colon TU_1 \to TU_2$ is the tangent map of λ. Since $T\colon \mathrm{Man} \to \mathrm{Man}$ is a functor (Man being the category of smooth manifolds), the relation $\sim$ is an equivalence relation. The equivalence class $[\pi, v]$ of one of the above tupel is called *formal orbifold tangent vector* and define the *set of all formal orbifold tangent vectors at p* $\mathcal{T}_pQ$.
Consider $x_1, x_2 \in \pi^{-1}(p)$, $(U, G, \pi) \in \mathcal{U}$. The isotropy subgroup G_{x_i} acts on $T_{x_i}U$ via the linear diffeomorphisms $\gamma.v := T_{x_i}\gamma.v$. Every $\gamma \in G$ is a self-embedding of orbifold charts, whence

$$(\pi, v) \sim (\pi, T\gamma.v) \quad \forall \gamma \in G. \tag{4.1.1}$$

Let $\tilde{v} \in T_{x_1}U/G_{x_1}$ be the equivalence class of $v \in T_{x_1}U$ for $x_1 \in \pi^{-1}(p)$. We obtain a bijective map

$$k_\pi^{x_1}\colon T_{x_1}U/G_{x_1} \to \mathcal{T}_pQ, k_\pi^{x_1}(\tilde{v}) := T\pi(v) := [\pi, v].$$

Endow $\mathcal{T}_pQ$ with the unique topology making the bijection $k_\pi^{x_1}$ a homeomorphism. The space $\mathcal{T}_pQ$, is called *tangent space of Q at p*. We claim that the topology on $\mathcal{T}_pQ$ does neither depend on the choice of charts nor on the preimage in any given chart. Choose some chart (U, G, π). As a first step, we prove that the topology does not depend on the choice of the preimage in this chart:

Step 1: Choose another $x_2 \in \pi^{-1}(p)$. There is some $\gamma \in G$ with $\gamma.x_1 = x_2$. The isotropy groups of x_1 and x_2 are thus conjugated via $\gamma.G_{x_1}\gamma^{-1} = G_{x_2}$. The derived actions of G_{x_i} on $T_{x_i}U,\ i = 1,2$ are conjugated via the linear isomorphism $T_{x_1}\gamma$, which induces a homeomorphism $\widetilde{T_{x_1}\gamma}\colon T_{x_1}U/G_{x_1} \to T_{x_2}U/G_{x_2}$. For $v \in T_{x_1}U$ let $\tilde{v}$ be its image in $T_{x_1}U/G_{x_1}$ and compute

$$(k_\pi^{x_2})^{-1} \circ (k_\pi^{x_1})(\tilde{v}) = (k_\pi^{x_2})^{-1}[\pi, v] \overset{(4.1.1)}{=} (k_\pi^{x_2})^{-1}[\pi, T_{x_1}\gamma.v] = \widetilde{T_{x_1}\gamma}(\tilde{v}).$$

Since $\widetilde{T_{x_1}\gamma}$ is a homeomorphism, so is $(k_\pi^{x_2})^{-1}k_\pi^{x_1}: T_{x_1}U/G_{x_1} \to T_{x_2}/G_{x_2}$. In conclusion the topology on $\mathcal{T}_pQ$ does not depend on the choice of $x_i \in \pi^{-1}(p)$, whence the index x_i of $k_\pi^{x_i}$ will be omitted.

Step 2: Consider another chart (W, H, ψ) with $p \in \psi(W)$ and pick $y \in \psi^{-1}(p)$. By compatibility of charts, there are open subsets $x \in V_U \subseteq U$, $y \in V_W \subseteq W$ and a change of charts homomorphism $\lambda\colon V_U \to V_W$ with $\lambda(x) = y$. Shrinking the open sets V_U, V_W, we may assume that $(V_U, G_x, \pi|_{V_U})$ is an orbifold chart and λ an open embedding of orbifold charts. This map conjugates the G_x-action on T_xU to the H_y-action on T_yW again inducing a homeomorphism $\widetilde{T_x\lambda}\colon T_xV_U/G_x \to T_yV_W/H_y$. As in step 1 a well-defined homeomorphism is given by

$$k_\psi \circ k_\pi^{-1}\colon T_xU/G_x \to T_y/H_y, \tilde{v} \mapsto \widetilde{T\lambda}(\tilde{v}).$$

Therefore the topology on $\mathcal{T}_pQ$ is independent of the choice of charts.

4.1.2 Remark Let (U, G, π) be an orbifold chart with $p \in \operatorname{Im}\pi$. The homeomorphism $\mathcal{T}_pQ \cong T_xU/G_x$ for $x \in \pi^{-1}(p)$ allows us to think of $\mathcal{T}_pQ$ as an orbifold. In particular locally (i.e. in a suitable chart of the manifold used to define the tangent space) the tangent space $\mathcal{T}_pQ$ may be identified with a convex cone. In contrast to angent spaces of manifolds, the tangent spaces of an orbifold will not be vector spaces. Nevertheless, each orbifold tangent space contains a zero element $0_p := [\pi, 0_x]$, where (U, G, π) is a chart with $p = \pi(x)$ and $0_x \in T_xU$ the zero element.
In the manifold case, our definition boils down to: The tangent space of a manifold (i.e. trivial orbifold) at p is the tangent space of the manifold at p.

4.1.3 Definition (Tangent Orbibundle) Consider the set $\mathcal{T}Q := \bigcup_{p\in Q} \mathcal{T}_pQ$. Since the tangent spaces are mutually disjoint, we derive a well-defined map

$$\pi_{\mathcal{T}Q}\colon \mathcal{T}Q \to Q, [\psi, v] \mapsto \psi(x), \text{ where } v \in T_x \operatorname{dom}\psi.$$

Let $(U, G, \psi) \in \mathcal{U}$ be an arbitrary chart, then G acts on TU via the derived action $\gamma.X := T\gamma(X)$. Define $\Pi\colon TU \to TU/G$ to be the quotient map to the orbit space with respect to this action. Using the notation of Construction 4.1.1 we obtain a map for $(U, G, \psi) \in \mathcal{U}$:

$$T\psi\colon TU \to \mathcal{T}Q, v \mapsto [\psi, v]$$

In particular, each $v \in T_xU$ is mapped to some $[\psi, v] \in \mathcal{T}_{\psi(x)}Q$. We equip $\mathcal{T}Q$ with the final topology with respect to the family $\{T\psi\}_{(U,G,\psi)\in\mathcal{U}}$.
This topology induces a canonical orbifold structure on $\mathcal{T}Q$. An atlas for this orbifold is given by the family $(TU, G, T\psi)$, where (U, G, ψ) runs through $\mathcal{U}$. The G-action of the chart $(TU, G, T\psi)$ is the derived action of G, i.e. $\gamma.v := T\gamma(v)$. With respect to this structure $\pi_{\mathcal{T}Q}$ induces an orbifold map. Its lifts are given by the bundle projections $TU \to U$, for $(U, G, \pi) \in \mathcal{U}$.
We define the *tangent orbibundle* (or *tangent orbifold*) $\mathcal{T}(Q, \mathcal{U})$ of $(Q, \mathcal{U})$. It is the unique orbifold derived from $(\mathcal{T}Q, \mathcal{B}_{\mathcal{T}\mathcal{U}})$, where $\mathcal{B}_{\mathcal{T}\mathcal{U}}$ is the maximal atlas induced by $\mathcal{T}\mathcal{U}$. A detailed proof for the details of this construction will be given in the next Lemma.

4.1.4 Lemma *Let $(Q,\mathcal{U})$ be an n-dimensional orbifold. Using the notation of Definition 4.1.3 the following statements hold:*

(a) *Let $(U,G,\psi),(V,H,\varphi) \in \mathcal{U}$ and $\lambda\colon U \supseteq W \to W' \subseteq V$ a change of charts. Its tangent map $T\lambda\colon TW \to TW'$ is a diffeomorphism with $T\varphi T\lambda = T\psi|_{TW}$.*
(b) *For any chart $(U,G,\psi) \in \mathcal{U}$ we set $\tilde{U} := \psi(U)$ and $T\tilde{U} := \operatorname{Im} T\psi$. Tehn $T\tilde{U}$ is an open set in $\mathcal{T}Q$ and $T\psi$ is an open map.*
(c) *$\mathcal{TU} := \{ (TU,G,T\psi) | (U,G,\psi) \in \mathcal{U} \}$ is an orbifold atlas for $\mathcal{T}Q$.*
(d) *The map $\pi_{\mathcal{T}Q}\colon \mathcal{T}Q \to Q, [\psi,v] \mapsto \psi(x),\ v \in T_xU$ is continuous and $\mathcal{T}Q$ is a Hausdorff paracompact space. In conclusion $\mathcal{T}(Q,\mathcal{U})$ is an orbifold.*
(e) *$\pi_{\mathcal{T}Q}$ induces a morphism of orbifolds $\pi_{\mathcal{T}(Q,\mathcal{U})} \in \mathbf{Orb}\,(\mathcal{T}(Q,\mathcal{U}),(Q,\mathcal{U}))$.*
(f) *The topology on $\mathcal{T}Q$ induces on each $\mathcal{T}_pQ$ the topology constructed in Construction 4.1.1.*

Proof. (a) For the change of charts λ, the tangent map $T\lambda\colon TW \to TW'$ is a diffeomorphism. It suffices to prove the commutativity for each element of T_rW, where $r \in W$ is arbitrary. Since λ is a change of charts, $\varphi\lambda = \psi|_{\operatorname{dom}\lambda}$ holds. The definition of $T_{\psi(r)}(Q,\mathcal{U})$ yields $[\psi,v] = [\varphi, T\lambda(v)]$. We obtain for $v \in T_rW$:

$$T\varphi T\lambda(v) = [\varphi, T\lambda(v)] = [\psi, v] = T\psi(v)$$

(b) The space $\mathcal{T}Q$ is endowed with the final topology with respect to the mappings $T\pi$. To prove the assertion we need to show that $(T\pi)^{-1}(T\psi(V))$ is an open set for every (W,H,π) and each $V \subseteq TU$ open. Define the *set of change of charts from U to W*:

$$\mathcal{C}h_{U,W} := \{ \lambda\colon U \supseteq \operatorname{dom}\lambda \to \operatorname{cod}\lambda \subset W | \lambda \text{ is a change of charts} \}$$

Then one computes its preimage as

$$\begin{aligned}(T\pi)^{-1}(T\psi(V)) &= \{ w \in TW | [\pi, w] \in T\psi(V) \} \\ &= \{ w \in TW | \exists \lambda \in \mathcal{C}h_{U,W}, w = T\lambda(v) \text{ for some } v \in TV \} \\ &= \bigcup_{\lambda \in \mathcal{C}h_{U,W}} T\lambda(\operatorname{dom}\lambda \cap V) \subseteq TW.\end{aligned}$$

Each $T\lambda$ is a diffeomorphism onto its (open) image, whose domain is an open set. Thus every $T\lambda(\operatorname{dom} T\lambda \cap V)$ is an open subset in TW. This proves $(T\pi)^{-1}(T\psi(V))$ to be an open set, whence $T\psi$ is an open map with open image $T\tilde{U}$ in $\mathcal{T}Q$.

(c) Let $(U,G,\phi) \in \mathcal{U}$ be an arbitrary chart, then $T\phi$ has an open image by (b). Consider the map $T\overline{\phi}\colon TU/G \to \operatorname{Im} T\phi, v \mapsto [\phi, v]$. Combining Proposition 2.2.2 with the definition of the equivalence relation in Construction 4.1.1, this map is a well-defined bijective map. We may factor $T\phi$ as $T\phi = T\overline{\phi} \circ \Pi$, where Π is the quotient map to the orbit space associated to the G action on TU. Since quotient maps and $T\phi$ are continuous, $T\overline{\phi}$ is continuous. Let $V \subseteq TU/G$ be an open set, then $\Pi^{-1}(V)$ is an open set. Since $T\phi$ is open by (b) the set $T\overline{\phi}(V) = T\phi\Pi^{-1}(V)$ is an open set. Thus $T\overline{\phi}$ is an open map and in conclusion $T\phi$ may be factored as the quotient map to the orbit space associated to the group action composed with a homeomorphism. In particular, the family

$$\mathcal{TU} := \{ (TU, G, T\pi) | (U, G, \pi) \in \mathcal{U} \}$$

is a family of orbifold charts on $\mathcal{T}Q$. In (a) we constructed a family of maps which are change of chart maps, for $\mathcal{TU}$. Using this family of change of charts, the definition of the tangent spaces $\mathcal{T}_pQ$ shows that each pair of orbifold charts in $\mathcal{TU}$ is compatible. Furthermore every element of $\mathcal{T}Q$ is contained within the image of some chart. Thus $\mathcal{TU}$ is an orbifold atlas inducing a unique orbifold structure of dimension $2 \cdot \dim(Q,\mathcal{U})$ on $\mathcal{T}(Q,\mathcal{U})$.

(d) The definitions of $\pi_{\mathcal{T}Q}$ and $\mathcal{T}Q$ together with the compatibility of orbifold charts yields $\pi_{\mathcal{T}Q}^{-1}(\psi(U)) = T\psi(TU)$, for every $(U,G,\psi) \in \mathcal{U}$. The images of orbifold charts in $\mathcal{U}$ form a basis of the topology by Lemma 2.4.1, whence $\pi_{\mathcal{T}Q}$ is continuous by [20, Proposition 1.4.1.].
The space $\mathcal{T}Q$ *is a Hausdorff space*: Let $x, y \in \mathcal{T}Q$ be distinct points.
First case: $\pi_{\mathcal{T}Q}(x) \neq \pi_{\mathcal{T}Q}(y)$. There are orbifold charts $(U_x, G_x, \psi_x), (U_y, G_y, \psi_y) \in \mathcal{U}$ such that $\pi_{\mathcal{T}Q}(x) \in \psi_x(U_x)$, $\pi_{\mathcal{T}Q}(y) \in \psi_y(U_y)$ and $\psi_x(U_x) \cap \psi_y(U_y) = \emptyset$ hold. As the images of those charts do not intersect, the set $\mathcal{C}h_{U_x,U_y}$ is empty. By construction of the equivalence relation, $T\psi_x(TU_x) \cap T\psi_y(TU_y) = \emptyset$. Hence $x \in \pi_{\mathcal{T}Q}^{-1}(\psi_x(U_x))$ and $y \in \pi_{\mathcal{T}Q}^{-1}(\psi_y(U_y))$ are contained in disjoint open sets.
Second case: $\pi_{\mathcal{T}Q}(x) = \pi_{\mathcal{T}Q}(y)$. Choose any orbifold chart (U,G,ψ) with $\pi_{\mathcal{T}Q}(x) \in \psi(U)$. Then $x, y \in \pi_{\mathcal{T}Q}^{-1}(\psi(U)) = T\psi(TU)$. Both x and y are contained in $T\psi(TU)$, which is homeomorphic to the orbit space TU/G. This space is Hausdorff by Lemma B.1.4 and there are disjoint open subsets $x \in V_x, y \in V_y$ of $T\psi(TU)$. As $T\psi(TU)$ is open, both points are contained in disjoint open subsets of $\mathcal{T}Q$. In conclusion the space $\mathcal{T}Q$ is a Hausdorff space.
The space $\mathcal{T}Q$ *is paracompact*: Connected components of $\mathcal{T}Q$ are closed, therefore [20, Theorem 5.1.35] implies that Q will be paracompact if each connected component of $\mathcal{T}Q$ is paracompact. We claim that each connected component C of $\mathcal{T}Q$ is second countable. If this were true, paracompactness of a component is assured by the following observations: The quotient map to an orbit space preserves locally compact spaces by Lemma B.1.4. Thus $\mathcal{T}Q$ is locally compact, whence a regular space. Combining [20, Theorem 3.8.1] and [20, Theorem 5.1.2] second countability of a component implies paracompactness of that component.
Proof of the claim: Every component $C' \subseteq Q$ is second countable (cf. Proposition 2.4.3). The continuous map $\pi_{\mathcal{T}Q}$ maps C into some component $C' \subseteq Q$. Since C' is second countable, there is a countable base $\mathcal{B}$ of the topology on C'. Orbifold charts with images in C' also form a base of the topology by Lemma 2.4.1. Thus without loss of generality $\mathcal{B}$ contains only (open) images of orbifold charts $(U_i, G_i, \pi_i) \in \mathcal{U}$. By construction of $\pi_{\mathcal{T}Q}$, the countable family of open sets $\mathcal{TB} := \{\, T\pi_i(TU_i) | (U_i, G_i, \pi_i) \in \mathcal{B} \,\}$ covers C. Observe that $T\tilde{U_i} \cong TU_i/G_i$ and TU_i is the tangent manifold of a connected paracompact manifold hence itself connected paracompact and thus second countable by Proposition 2.4.2. The quotient map to the orbit space is continuous and open by Lemma B.1.4 which implies that $T\tilde{U_i}$ is also second countable. As a countable union of open and second countable spaces, the component C is second countable.

(e) The map $\pi_{\mathcal{T}Q}$ is continuous by (d) and we have to construct lifts for $\pi_{\mathcal{T}Q}$: Consider an arbitrary orbifold chart $(TU, G, T\psi) \in \mathcal{TU}$. Let $\pi_{TU} \colon TU \to U$ be the bundle projection of the tangent bundle. This map is smooth (as a map of smooth manifolds), and we obtain a commutative diagram:

$$\begin{tikzcd} TU \arrow[r, "T\psi"] \arrow[d, "\pi_{TU}"] & T\tilde{U} \arrow[d, "\pi_{\mathcal{T}Q}"] \\ U \arrow[r, "\psi"] & \tilde{U} \end{tikzcd}$$

Define the quasi-pseudogroup $P_{\pi_{\mathcal{T}Q}} := \bigcup_{(U,W)\in\mathcal{U}\times\mathcal{U}} \{ T\lambda | \lambda \in \mathcal{C}h_{U,W} \}$. We have to show that this quasi-pseudogroup generates $\mathbf{\Psi}(\mathcal{TU})$. Let $\varphi \in \mathbf{\Psi}(\mathcal{TU})$ and pick an arbitrary $v \in \operatorname{dom}\varphi$. Then there are $(TU, G, T\pi), (TV, H, T\psi) \in \mathcal{TU}$ and an open set $v \in \Omega \subseteq TU$, such that $\varphi|_\Omega$ is a diffeomorphism onto an open set $\Omega' \subseteq TV$ which contains $w := \varphi(v)$. Since $T\psi(w) = T\pi(v)$ holds, the equivalence relation shows that there are open sets $x \in W \subseteq U$, $y \in W' \subseteq V$ and a change of charts morphism $\lambda \colon W \to W'$ such that $v \in T_xW$, $w \in T_yW'$ and $T\lambda(v) = w$. Shrinking W and W' we may assume, that $T\lambda \colon TW \to TW'$ is an open embedding of orbifold charts. Thus on TW the maps $T\lambda$ and $\varphi|_{TW}$ are embeddings of orbifold charts. By Proposition 2.2.2, there is a $h \in H_w$ such that $h.T\lambda = \varphi|_{TW}$. The definition of the group action on charts in $\mathcal{TU}$ yields $\varphi|_{TW} = h.T\lambda = T(h.\lambda)$. Thus $h.\lambda \in \mathbf{\Psi}(\mathcal{U})$ implies$T(h.\lambda) \in P_{\pi_{\mathcal{T}(Q,\mathcal{U})}}$. In conclusion $P_{\pi_{\mathcal{T}Q}}$ generates $\mathbf{\Psi}(\mathcal{TU})$. Define the map

$$\nu_{\pi_{\mathcal{T}Q}} \colon P_{\pi_{\mathcal{T}Q}} \to \mathbf{\Psi}(\mathcal{U}), T\lambda \mapsto \lambda.$$

By construction this map satisfies (R4a)-(R4d) of definition E.2.3 and therefore

$$\pi_{\mathcal{T}(Q,\mathcal{U})} := (\pi_{\mathcal{T}Q}, \{ \pi_{TU} | (U, G, \pi) \in \mathcal{U} \}, [P_{\pi_{\mathcal{T}Q}}, \nu_{\pi_{\mathcal{T}Q}}]) \in \operatorname{Orb}(\mathcal{TU}, \mathcal{U})$$

is a representative of an orbifold map. We call $\pi_{\mathcal{T}(Q,\mathcal{U})}$ the *bundle projection of the tangent orbibundle*. By abuse of notation, we let $\pi_{\mathcal{T}(Q,\mathcal{U})}$ be its equivalence class in $\mathbf{Orb}(\mathcal{T}(Q,\mathcal{U}), (Q,\mathcal{U}))$.

(f) Choose some orbifold chart $(U, G, \psi) \in \mathcal{U}$ such that $\pi_{\mathcal{T}Q}(p) \in \psi(U)$. Shrinking the chart, we may assume $\{ z \} = \psi^{-1}(\pi_{\mathcal{T}Q}(p))$, i.e. $G \cong \Gamma_{\pi_{\mathcal{T}Q}(p)}$. By construction $\mathcal{T}_pQ \subseteq T\psi(TU)$ holds. Recall from (c) that $T\psi = T\overline{\psi} \circ \Pi$, where Π is the quotient map to the orbit space with respect to the G-action on TU and $T\overline{\psi}$ is a homeomorphism. Observe $(T\overline{\psi})^{-1}(\mathcal{T}_pQ) = \Pi(T_zU)$. Notice that for manifolds the subspace topology of $T_zU \subseteq TU$ coincides with the usual topology of T_zU. As the quotient map to an orbit space is open, [19, VI. Theorem 2.1] proves that the subspace topology of $(T\overline{\psi})^{-1}(\mathcal{T}_pQ)$ and the quotient topology on $\Pi(T_zU) = T_zU/G$ coincide. In construction 4.1.1 $\mathcal{T}_pQ$ has been endowed with precisely the same topology. Hence the induced topology on $\mathcal{T}_pQ$ coincides with the one from definition 4.1.1.

□

Notice that for any trivial orbifold (i.e. for a manifold), the tangent orbibundle coincides with the tangent bundle of the manifold. For a non-trivial orbifold, an explicit example of a tangent orbifold will be computed in Example 4.3.8.

Mappings into the tangent orbifold admit representatives, which are charted maps whose range atlas is contained in $\mathcal{TU}$. Thus orbifold maps into the tangent orbifold always posses representatives which may be computed in the canonical orbifold charts of the tangent orbifold.

4.1.5 Lemma *Let $[\hat{f}] \in \mathbf{Orb}\,((Q,\mathcal{U}), \mathcal{T}(Q,\mathcal{U}))$ be an arbitrary orbifold map. There is a representative $\hat{f} \in [\hat{f}]$, such that the range atlas of $\hat{f}$ is contained in $\mathcal{TU}$. In other words $\hat{f}$ is a charted orbifold map with $\hat{f} \in \operatorname{Orb}(\mathcal{V}, \mathcal{TU})$, where $\mathcal{V}$ is some representative of $\mathcal{U}$.*

Proof. Let $[\hat{f}]$ be as above.Consider the bundle projection $\pi_{\mathcal{T}(Q,\mathcal{U})}$ from Lemma 4.1.4. We may thus compose $\pi_{\mathcal{T}(Q,\mathcal{U})} \circ [\hat{f}]$. Reviewing Lemma [51, Lemma 5.17], the composition in **Orb** is induced by

the composition of representatives of the equivalence classes. Hence there are representatives $\mathcal{V}, \mathcal{V}''$ of $\mathcal{U}$ resp. a representative $\mathcal{V}'$ of $\mathcal{B}_{\mathcal{TU}}$, together with charted orbifold maps $\hat{g} \in \mathrm{Orb}(\mathcal{V}, \mathcal{V}')$ such that $\hat{g} \in [\hat{f}]$ and $\hat{h} \in \mathrm{Orb}(\mathcal{V}', \mathcal{V}'')$ induced by the representative $\pi_{\mathcal{T}(Q,\mathcal{U})}$. (i.e. the composition is given as $\pi_{\mathcal{T}(Q,\mathcal{U})} \circ [\hat{f}] = [\hat{h} \circ \hat{g}]$). Let $\mathcal{V}'$ be indexed by J. Again by the proof of [51, Lemma 5.17], for every chart $(U_j', G_j', \psi_j') \in \mathcal{V}'$, there is an open embedding $\lambda_j \colon (U_j', G_j', \psi_j') \to (TV, H, T\pi)$ for some chart $(TV, H, T\pi) \in \mathcal{TU}$. Let $\mathcal{V}$ be indexed by I, then $\hat{g} = (f, \{ g_i | i \in I \}, [P_{\hat{g}}, \nu_{\hat{g}}])$ where $g_i \colon U_i \to U'_{\alpha(i)}$. We define a new set of lifts via $f_i := \lambda_{\alpha(i)} g_i$. Set $P_{\hat{f}} := P_{\hat{g}}$ and let

$$\nu_{\hat{f}} \colon P_{\hat{g}} = P_{\hat{f}} \to \mathbf{\Psi}(\mathcal{TU}), \gamma \mapsto \lambda_{\alpha(j)} \nu_{\hat{g}}(\gamma) \lambda_{\alpha(i)}^{-1}, \ \mathrm{dom}\, \gamma \subseteq U_j, \ \mathrm{cod}\, \gamma \subseteq U_i$$

Clearly each $\lambda_{\alpha(j)} \nu_{\hat{g}}(\gamma) \lambda_{\alpha(i)}^{-1}$ is a well-defined change of charts map in $\mathbf{\Psi}(\mathcal{TU})$ and $\nu_{\hat{f}}$ satisfies the requirements (R4a)-(R4d). Therefore $\hat{f} := (f, \{ f_i \}_{i \in I}, [P_{\hat{f}}, \nu_{\hat{f}}]) \in \mathrm{Orb}(\mathcal{V}, \mathcal{TU})$ is a representative of $[\hat{f}]$ with the desired properties. □

4.1.6 Remark

(a) Let $\mathcal{V}$ be a representative of the maximal atlas $\mathcal{U}$ of an orbifold $(Q,\mathcal{U})$. The group action on a chart in $\mathcal{V}$ acts on the tangent chart via the derived action. Since the tangent functor $T \colon \mathrm{Man} \to \mathrm{Man}$ (where Man is the category of smooth (not necessarily finite dimensional) manifolds) is functorial, Proposition 2.2.2 (e) and the definition of the tangent manifold imply that $\mathcal{T}\mathbf{\Psi}(\mathcal{V}) := \{ T\lambda | \lambda \in \mathbf{\Psi}(\mathcal{V}) \}$ is a quasi-pseudogroup which generates $\mathbf{\Psi}(\mathcal{TV})$. Furthermore, if P is some quasi-pseudogroup which generates $\mathbf{\Psi}(\mathcal{V})$, the quasi-pseudogroup $\mathcal{T}P := \{ T\lambda | \lambda \in P \}$ generates $\mathbf{\Psi}(\mathcal{TV})$.

(b) Let $\lambda, \mu \in \mathcal{C}h_{V,W}$ be change of charts and $X \in \mathrm{dom}\, T\lambda \cap \mathrm{dom}\, T\mu$ such that $\mathrm{germ}_X T\lambda = \mathrm{germ}_X T\mu$ holds. Furthermore let $U_X \subseteq TV$ be an open X neighborhood with $T\lambda|_{U_X} = T\mu|_{U_X}$. This implies $\lambda \pi_{TV}|_{U_X} = \mu \pi_{TV}|_{U_X}$. Since π_{TV} is an open map, $\pi_{TV}(U_X)$ is open and contains $\pi_{TV}(X)$, thus $\mathrm{germ}_{\pi_{TV}(X)} \lambda = \mathrm{germ}_{\pi_{TV}(X)} \mu$ holds.

The observation made in the last remark, allows us to introduce tangent orbifold maps:

4.1.7 Definition Let $(Q_i, \mathcal{U}_i)$, $i = 1, 2$ be orbifolds and $[\hat{f}] \in \mathbf{Orb}[(Q_1,\mathcal{U}_1), (Q_2,\mathcal{U}_2)]$ be a morphism with representative $\hat{f} = (f, \{ f_i \}_{i \in I}, [P, \nu]) \in \mathrm{Orb}(\mathcal{V}, \mathcal{W})$.
Furthermore let $\mathcal{V} := \{ (V_i, G_i, \psi_i) | i \in I \}$ and $\mathcal{W} := \{ (W_j, H_j, \varphi_j) | j \in J \}$. For two change of charts $T\lambda = T\mu$ is satisfied iff $\lambda = \mu$, whence $\mathcal{T}\nu \colon \mathcal{T}P \to \mathbf{\Psi}(\mathcal{TW}), T\lambda \mapsto T\nu(\lambda)$ is a well defined map. Here $\mathcal{T}P$ is the quasi-pseudogroup of some (P, ν) in the class $[P, \nu]$ as in remark 4.1.6 (a). The class $[\mathcal{T}P, \mathcal{T}\nu]$ does not depend on the choice of (P, ν) in $[P, \nu]$ by the definition of equivalence (cf. Definition E.2.5).
Combining remark 4.1.6 (b) and the properties (R4a)-(R4d) of Definition E.2.3 for the map ν w.r.t. $F := \coprod_{i \in I} f_i$, $\mathcal{T}\nu$ satisfies properties (R4a)-(R4d) with respect to $F' := \coprod_{i \in I} Tf_i$. In particular we derive $T\varphi_{\alpha(i)} Tf_i(T\lambda.x) = T\varphi_{\alpha(j)} Tf_j(x)$ for each $\lambda \in \mathcal{C}h_{V_j, V_i}$. Thus there is a well-defined continuous map $\mathcal{T}f \colon \mathcal{T}Q_1 \to \mathcal{T}Q_2, \mathcal{T}f(x) := T\varphi_{\alpha(i)} Tf_i T\psi_i^{-1}(x), \ x \in \mathrm{Im}\, T\psi_i$.

In conclusion, a charted map of orbifolds is given by

$$\widehat{\mathcal{T}f} := (\mathcal{T}f, \{\, Tf_i | i \in I \,\}, [\mathcal{T}P, \mathcal{T}\nu]) \in \mathrm{Orb}(\mathcal{T}\mathcal{V}, \mathcal{T}\mathcal{W})$$

The map $\widehat{\mathcal{T}f}$ is a representative of the *orbifold tangent map* $[\widehat{\mathcal{T}f}]$ of $[\hat{f}]$. We have to check that the construction of this map is functorial.

4.1.8 Lemma *The assignment* $\mathcal{T}\colon$ **Orb** $\rightarrow$ **Orb**, $(Q,\mathcal{U}) \mapsto \mathcal{T}(Q,\mathcal{U}), [\hat{f}] \mapsto [\widehat{\mathcal{T}f}]$ *is a functor, i.e.*

(a) *If* $\hat{\varepsilon} = (\mathrm{id}_Q, \{\, f_i \,\}_{i\in I}, [P,\nu]) \in \mathrm{Orb}(\mathcal{V},\mathcal{W})$ *is a lift of the identity* $\mathrm{id}_{(Q,\mathcal{U})}$, *then* $\widehat{\mathcal{T}\varepsilon}$ *is a lift of the identity* $\mathrm{id}_{\mathcal{T}(Q,\mathcal{U})}$

(b) *Let* $\hat{f} = (f, \{\, f_i | i \in I \,\}, [P_f, \nu_f]) \in \mathrm{Orb}(\mathcal{V},\mathcal{W})$ *and* $\hat{g} = (g, \{\, g_j | j \in J \,\}, [P_g, \nu_g]) \in \mathrm{Orb}(\mathcal{W},\mathcal{U})$ *then* $\widehat{\mathcal{T}g \circ f} = \widehat{\mathcal{T}g} \circ \widehat{\mathcal{T}f}$ *holds.*

(c) *Two representatives* $\hat{f}_1, \hat{f}_2$ *of* $[\hat{f}] \in$ **Orb** $((Q_1,\mathcal{U}_1),(Q_2,\mathcal{U}_2))$ *induce equivalent charted orbifold maps, i.e.* $[\widehat{\mathcal{T}f_1}] = [\widehat{\mathcal{T}f_2}]$. *The map* $[\widehat{\mathcal{T}f}]$ *is called* orbifold tangential map.

(d) $[\widehat{\mathcal{T}g \circ f}] = [\widehat{\mathcal{T}g}] \circ [\widehat{\mathcal{T}f}]$ *holds for* $[\hat{f}] \in$ **Orb** $((Q_1,\mathcal{U}_1),(Q_2,\mathcal{U}_2))$, $[\hat{g}] \in$ **Orb** $((Q_2,\mathcal{U}_2),(Q_3,\mathcal{U}_3))$.

Proof. (a) For each $i \in I$ let the lifts $f_i\colon V_i \rightarrow W_{\alpha(i)}$ be given with respect to the charts (V_i, G_i, ψ_i) and $(W_{\alpha(i)}, H_{\alpha(i)}, \varphi_{\alpha(i)})$. Here $\alpha\colon I \rightarrow J$ is the map which assigns to f_i the chart $W_{\alpha(i)}$. Each f_i is a local diffeomorphism by Definition E.3.5. Using functoriality of T, again Tf_i is a local diffeomorphism. By Proposition E.5.3 the assertion will be true if $\mathcal{T}\,\mathrm{id}_Q = \mathrm{id}_{\mathcal{T}Q}$ holds. Consider $x \in \mathcal{T}Q$ with $x \in \mathrm{Im}\, T\psi_i$ for some $i \in I$. Choose $z_x \in TV_i$ with $T\psi_i(z_x) = x$ and observe that by Proposition E.3.2, we may choose orbifold charts $(S_x, G_x, \psi_x|_{S_x})$ and $(S'_x, G'_x, \psi_x|_{S_x})'$ with $\pi_{TV_i}(x) \in S_x$ such that f_i induces the identity on S_x with respect to id_{S_x} and $(f_i|_{S_x})^{-1}$. Hence $f_i|_{S_x}$ is a change of charts map, which implies $\mathcal{T}\,\mathrm{id}_Q(x) = \mathcal{T}\,\mathrm{id}_Q(T\psi_i(z_x)) = T\varphi_{\alpha(i)} Tf_i(z_x) = T\varphi_{\alpha(i)} T(f_i|_{S_x})(z_x) = x$.

(b) Define $h_i := g_{\alpha(i)} \circ f_i$ and $h = g \circ f$. Then $\hat{g} \circ \hat{f}$ is given by $\hat{h} = (h, \{\, h_i | i \in I \,\}, [P_h, \nu_h])$. From Definition 4.1.7 we infer $\widehat{\mathcal{T}(g \circ f)} = (\mathcal{T}h, \{\, Th_i | i \in I \,\}, [\mathcal{T}P_h, \mathcal{T}\nu_h])$.
By construction one has $\widehat{\mathcal{T}f} \in \mathrm{Orb}(\mathcal{T}\mathcal{V}, \mathcal{T}\mathcal{W})$ and $\widehat{\mathcal{T}g} \in \mathrm{Orb}(\mathcal{T}\mathcal{W}, \mathcal{T}\mathcal{U})$. These charted orbifold maps may therefore be composed as in construction E.4.1: The charted orbifold map $\widehat{\mathcal{T}f} \circ \widehat{\mathcal{T}g}$ is given as $\hat{h}_{\mathcal{T}} := (\mathcal{T}g \circ \mathcal{T}f, \left\{\, Tg_{\alpha(i)} \circ Tf_i \middle| i \in I \,\right\}, [P_{h_{\mathcal{T}}}, \nu_{h_{\mathcal{T}}}])$. By functoriality of T we have $h_i = T(g_{\alpha(i)} \circ f_i) = Tg_{\alpha}(i) Tf_i$ for $i \in I$. Hence the lifts of $\widehat{\mathcal{T}(g \circ f)}$ and $\hat{h}_{\mathcal{T}}$ coincide for each $i \in I$. We conclude $\mathcal{T}h = \mathcal{T}g \circ \mathcal{T}f$.
If $(\mathcal{T}P_h, \mathcal{T}\nu_h) \sim (P_{h_{\mathcal{T}}}, \nu_{h_{\mathcal{T}}})$ holds, both maps will be equivalent as charted orbifold maps. By construction of the quasi-pseudogroups this indeed follows directly from the functoriality of T and property (R4b) of Definition E.2.3. However, since quasi-pseudogroups work with the germs of maps, the computation has to be carried out on the germ level. Here are the technical details:
Let $\lambda, \mu \in \mathcal{C}h_{V_i,V_j}, i,j \in I$, $\lambda \in \mathcal{T}P_h$, $\mu \in P_{h_{\mathcal{T}}}$ and $X \in \mathrm{dom}\,\lambda \cap \mathrm{dom}\,\mu$ with $\mathrm{germ}_X\,\lambda = \mathrm{germ}_X\,\mu$. To assert the equivalence, we have to prove the identity

$$\mathrm{germ}_{Th_i(X)}\,\mathcal{T}\nu_h(\lambda) = \mathrm{germ}_{Th_i(X)}\,\nu_{h_{\mathcal{T}}}(\mu). \tag{4.1.2}$$

Set $x := \pi_{TV_i}(X)$. By definition of the quasi-pseudogroups of $\hat{f}$ and $\hat{g}$ (combine Remark 4.1.6 and Construction E.4.1), we obtain the following data:

1. $\eta, \rho \in P_f$, $x \in U_{\eta,x}, U_{\rho,x}$ open and $\eta|_{U_{\eta,x}}, \rho|_{U_{\rho,x}} \in P_h$ with $\lambda = T\eta|_{U_{\eta,x}}$ and $\operatorname{germ}_X \mu = \operatorname{germ}_X T\rho$,
2. $\xi_{\eta,x}, \xi_{\rho,x} \in P_g$ with $\nu_h(\eta|_{U_{\eta,x}}) = \nu_g(\xi_{\eta,x})$ and $\operatorname{germ}_{f_i(x)} \xi_{\eta,x} = \operatorname{germ}_{f_i(x)} \nu_f(\eta)$, respectively for $\nu_h(\rho|_{U_{\rho,x}}) = \nu_g(\xi_{\rho,x})$ and $\operatorname{germ}_{f_i(x)} \xi_{\rho,x} = \operatorname{germ}_{f_i(x)} \nu_f(\rho)$
3. $\xi_{\mu,X} \in \mathcal{T}P_g$ with $\nu_{h_\mathcal{T}}(\mu) = \mathcal{T}\nu_g(\xi_{\mu,X})$ and $\operatorname{germ}_{Tf_i(X)} \xi_{\mu,X} = \operatorname{germ}_{Tf_i(X)} \mathcal{T}\nu_f(T\rho)$.

For $\phi, \psi \in P_f$ and $z \in \operatorname{dom}\phi \cap \operatorname{dom}\psi$ remark 4.1.6 (b) implies $\operatorname{germ}_z \phi = \operatorname{germ}_z \psi$ if and only if $\operatorname{germ}_X T\phi = \operatorname{germ}_X T\psi$ for some $X \in T_zV_i$. Exploiting property (R4b) for ν_f we obtain $\operatorname{germ}_{f_i(x)} \nu_f(\phi) = \operatorname{germ}_{f_i(x)} \nu_f(\psi)$, whence $\operatorname{germ}_{Tf_i(X)} \mathcal{T}\nu_f(T\phi) = \operatorname{germ}_{Tf_i(X)} \mathcal{T}\nu_f(T\psi)$ holds. Analogously the same holds for ν_g and ν_h by 1. and 2.:

$$\operatorname{germ}_{Th_i(X)} \mathcal{T}\nu_h(\lambda) = \operatorname{germ}_{Th_i(X)} T\nu_h(\eta|_{U_{\eta,x}}) = \operatorname{germ}_{Th_i(X)} T\nu_g(\xi_{\eta,x})$$

We already know $\operatorname{germ}_X T\eta = \operatorname{germ}_X \lambda = \operatorname{germ}_X \mu = \operatorname{germ}_X T\rho$ and by remark 4.1.6 (b) $\operatorname{germ}_x \eta = \operatorname{germ}_x \rho$ follows. Using property (R4b) for ν_f and 2. one obtains $\operatorname{germ}_{f_i(x)} \xi_{\eta,x} = \operatorname{germ}_{f_i(x)} \nu_f(\eta) = \operatorname{germ}_{f_i(x)} \nu_f(\rho)$.
Together with 3. this yields $\operatorname{germ}_{Tf_i(X)} T\xi_{\eta,x} = \operatorname{germ}_{Tf_i(X)} \mathcal{T}\nu_f(T\rho) = \operatorname{germ}_{Tf_i(X)} \xi_{\mu,X}$. Again by 3. and property (R4b) for $\mathcal{T}\nu_g$ we derive:

$$\operatorname{germ}_{Th_i(X)} \mathcal{T}\nu_h(\lambda) = \operatorname{germ}_{Th_i(X)} \mathcal{T}\nu_g(T\xi_{\eta,x}) = \operatorname{germ}_{Th_i(X)} \mathcal{T}\nu_g(\xi_{\mu,X}) = \operatorname{germ}_{Th_i(X)} \nu_{h_\mathcal{T}}(\mu)$$

Since X, λ, μ were arbitrary, $(\mathcal{T}P_h, \mathcal{T}\nu_h) \sim (P_{h_\mathcal{T}}, \nu_{h_\mathcal{T}})$ holds and we conclude $\widehat{\mathcal{T}(g \circ f)} = \widehat{\mathcal{T}g} \circ \widehat{\mathcal{T}f}$.

(c) We already know by (a) and (b) that $\mathcal{T}$ is functorial on charted orbifold maps. Hence we may apply $\mathcal{T}$ to the diagramm (E.4.2) which defines the equivalence of charted orbifold maps (cf. Definition E.4.3 and the assertion follows.

(d) This is just the combination of (b) and (c).

□

4.2. Orbisections

In this section we study sections of an orbifold into its tangent orbibundle. These maps will be called "orbisections" and may be thought of as an analogon to the vector fields on manifolds. In this section $(Q,\mathcal{U})$ will be an orbifold.

4.2.1 Definition A map of orbifolds $[\hat{\sigma}] \in \mathbf{Orb}((Q,\mathcal{U}),\mathcal{T}(Q,\mathcal{U}))$ is called *orbisection* if it satisfies

$$\pi_{\mathcal{T}(Q,\mathcal{U})} \circ [\hat{\sigma}] = \mathrm{id}_{(Q,\mathcal{U})}$$

Its *support* $\mathrm{supp}[\hat{\sigma}]$ is the closure of $\{ x \in Q | \sigma(x) \neq 0_x \}$, where $0_x \in \mathcal{T}_x Q$ is the zero-element. We define the *set of all orbisections* $\mathfrak{X}_{\mathrm{Orb}}(Q)$ of the orbifold $(Q,\mathcal{U})$.
An orbisection $[\hat{\sigma}] \in \mathfrak{X}_{\mathrm{Orb}}(Q)$ with $\mathrm{supp}[\hat{\sigma}] \subseteq K$ for some compact subset $K \subseteq Q$, is called *compactly supported (in K)*.
For $K \subseteq Q$ compact define the set $\mathfrak{X}_{\mathrm{Orb}}(Q)_K := \{ [\hat{\sigma}] \in \mathfrak{X}_{\mathrm{Orb}}(Q) | \mathrm{supp}[\hat{\sigma}] \subseteq K \}$ of *orbisections supported in K*. Let $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$ be the *set of all compactly supported orbisections in $\mathfrak{X}_{Orb}(Q)$*.

If M is a trivial orbifold (i.e. a manifold), orbisections are vector fields on the manifold. In this case, it is often advantageous to consider the representative of a vector field $X \colon M \to TM$ in pairs of charts. For a manifold chart Ψ of M, this representative is defined to be $X_\Psi := T\Psi \circ X \circ \Psi^{-1}$. These representatives are invaluable for the study of vector fields. It is possible to obtain lifts of a similar kind for orbisections on arbitrary orbifolds.

4.2.2 Definition Consider $[\hat{\sigma}] \in \mathfrak{X}_{\mathrm{Orb}}(Q)$ with $\hat{\sigma} := (\sigma, \{ \sigma_i \}_{i \in I}, P_\sigma, \nu_\sigma) \in \mathrm{Orb}(\mathcal{V}, \mathcal{T}\mathcal{V})$. If every σ_i is a map $\sigma_i \colon V_i \to TV_i$, such that $\pi_{TV_i} \circ \sigma_i = \mathrm{id}_{V_i}$ holds, the family $\{ \sigma_i \}_{i \in I}$ is called *family of canonical lifts for the orbisection $[\hat{\sigma}]$ with respect to $\mathcal{V}$*. If there is no risk of confusing which orbifold atlas is meant, we will also say that $\{ \sigma_i \}_{i \in I}$ is a *canonical family* for $[\hat{\sigma}]$.

Representatives of orbisections with canonical lifts with respect to a given atlas are unique:

4.2.3 Lemma *Let $[\hat{f}] \in \mathfrak{X}_{Orb}(Q)$ and $\mathcal{V} \subseteq \mathcal{U}$ be an arbitrary orbifold atlas, such that there exists a representative $\hat{h} = (f, \{ f_i \}_{i \in I}, P_h, \nu_h) \in \mathrm{Orb}(\mathcal{V}, \mathcal{T}\mathcal{V})$ whose lifts form a canonical family for $[\hat{f}]$. Then $\hat{h}$ is unique, i.e. if there is another representative of $[\hat{f}]$ whose lifts form a canonical family with respect to $\mathcal{V}$, then the members of this family must coincide with $\{ f_i \}_{i \in I}$.*

Proof. Let $\hat{g} = (f, \{ g_i | i \in I \}, P_g, \nu_g) \in \mathrm{Orb}(\mathcal{V}, \mathcal{T}\mathcal{V})$ be another representative of $[\hat{f}]$ whose lifts form a canonical family with respect to $\mathcal{V}$. For each chart (V_i, G_i, ψ_i), $i \in I$ we have $\pi_{TV_i} f_i = \pi_{TV_i} g_i$. On the other hand g_i and f_i are lifts of f, thus for every point $x \in V_i$, there is $\gamma_x \in G_i$, such that $T\gamma_x f_i(x) = \gamma_x.f_i(x) = g_i(x)$. Combining these observations, we derive

$$x = \pi_{TV_i} f_i(x) = \pi_{TV_i} g_i(x) = \pi_{TV_i} T\gamma_x f_i(x) = \gamma_x.x \tag{4.2.1}$$

Thus for each $x \in V_i \setminus \Sigma_{G_i}$ (i.e. x is non-singular), we derive $\gamma_x = \mathrm{id}_{V_i}$ and $f_i(x) = g_i(x)$. The continuous maps f_i and g_i coincide on the dense set $V_i \setminus \Sigma_{G_i}$ and therefore $f_i = g_i$. $\square$

It turns out that analogous to vector fields on manifolds, one is able to construct a canonical family for each orbisection with respect to any given orbifold atlas. At first we have to assure that there is at least some canonical lift for a given orbisection:

4.2.4 Lemma *For every orbisection $[\hat{f}] \in \mathfrak{X}_{Orb}(Q)$ there is a representative $\mathcal{V}$ of $\mathcal{U}$ indexed by some I and a representative of an orbifold map $\hat{g} = (f, \{ f_i \}, [P_{\hat{g}}, \nu_{\hat{g}}]) \in \mathrm{Orb}(\mathcal{V}, \mathcal{TV})$, such that*

(a) $\hat{g} \in [\hat{f}]$,
(b) $\{ f_i \}_{i \in I}$ *is a canonical family for* $[\hat{f}]$ *with respect to* $\mathcal{V}$.

Proof. Following Lemma 4.1.5, we choose an orbifold atlas $\mathcal{W}$ indexed by I, such that there is a representative $\hat{h} = (f, \{ h_i \}_{i \in I}, [P_{\hat{h}}, \nu_{\hat{h}}]) \in \mathrm{Orb}(\mathcal{W}, \mathcal{TU})$ of $[\hat{f}]$. For $i \in I$, $(V_i, G_i, \psi_i) \in \mathcal{W}$ and $(TU_{\alpha(i)}, G_{\alpha(i)}, \pi_{\alpha(i)}) \in \mathcal{TU}$ its lifts are given as $h_i \colon V_i \to TU_{\alpha(i)}$. By Lemma 4.1.5 the composition $h_i^1 := \pi_{TU_{\alpha(i)}} \circ h_i \colon V_i \to U_{\alpha(i)}$ is a local lift of id_Q, since $\pi_{\mathcal{T}(Q,\mathcal{U})} \circ [\hat{h}] = \mathrm{id}_{(Q,\mathcal{U})}$. For each $v \in V_i$ there is an open G_i-stable set V_i^v by Proposition E.3.2, such that $h_i^1|_{V_i^v}$ is an open embedding of orbifold charts.
Thus V_i may be covered by a family $\left\{ V_i^j \middle| j \in J_i \right\}$ of open G_i-stable subsets, such that $h_i^1|_{V_i^j}$ is an embedding of the orbifold chart $(V_i^j, G_{V_i^j}, \psi_i|_{V_i^j})$ into $W_{\alpha(i)}$. Define an orbifold atlas $\mathcal{V} \subseteq \mathcal{U}$ via $\mathcal{V} := \left\{ (V_i^j, G_{V_i^j}, \psi_i|_{V_i^j}) \middle| i \in I, j \in J_i \right\}$. Since h_i^1 is invertible on each V_i^j, $j \in J_i$ one may construct a family of lifts for f as follows: Set

$$f_i^j := T(h_i^1|_{V_i^j})^{-1} \circ h_i|_{V_i^j} \colon V_i^j \to TV_i^j.$$

Since h_i^1 is a local lift of the identity, so is Th_i^1 (cf. Lemma 4.1.8 (a)). The f_i^j are induced by h_i with respect to the inclusion of V_i^j and the open embedding $Th_i^1|_{TV_i^j}$. Thus f_i^j is a lift of f, such that $\pi_{TV_i^j} \circ f_i^j = \mathrm{id}_{V_i^j}$ holds. The members of $\mathcal{F} := \left\{ f_i^j \middle| i \in I, j \in J_i \right\}$ lift f with respect to $\mathcal{V}$.
We proceed to construct a pair (P, ν), such that $\hat{f} := (f, \mathcal{F}, [P, \nu]) \in \mathrm{Orb}(\mathcal{V}, \mathcal{TV})$ is a representative of $[\hat{f}]$. Choose a representative $(P_{\hat{f}}, \nu_{\hat{f}})$ of $[P_{\hat{h}}, \nu_{\hat{h}}]$. There are well-defined maps $s, t \colon P_{\hat{f}} \to I$, such that for each $\lambda \in P_{\hat{f}}$ we have $\operatorname{dom} \lambda \subseteq V_{s(\lambda)}$ and $\operatorname{cod} \lambda \subseteq V_{t(\lambda)}$. The sets $V_{s(\lambda)}, V_{t(\lambda)}$ are covered by the open sets $V_{s(\lambda)}^j, j \in J_{s(\lambda)}$ resp. $V_{t(\lambda)}^k, k \in K_{t(\lambda)}$. For every $\lambda \in P_{\hat{g}}$ using corestriction and restriction of λ, we define a family $\left\{ \mu_k^j(\lambda) \middle| j \in J_{s(\lambda)}, k \in K_{t(\lambda)} \right\}$ of change of charts morphisms via:

$$\mu_j^k(\lambda) := \lambda|_{\lambda^{-1}(V_{t(\lambda)}^k) \cap V_{s(\lambda)}^j}^{V_{t(\lambda)}^k} : \lambda^{-1}(V_{t(\lambda)}^k) \cap V_{s(\lambda)}^j \to V_{t(\lambda)}^k$$

We claim that $P := \left\{ \mu_j^k(\lambda) \middle| \lambda \in P_{\hat{f}}, j \in J_{s(\lambda)}, k \in K_{t_\lambda} \right\}$ is a quasi-pseudogroup generating $\mathbf{\Psi}(\mathcal{V})$: Observe that $P_{\hat{h}}$ is a quasi-pseudogroup generating $\mathbf{\Psi}(\mathcal{W})$ and P was constructed by restricting

elements of $P_{\hat{h}}$ to open sets. Each chart $(V_i^j, G_{V_i^j}, \psi_i|_{V_i^j})$ is embedded as an open set in V_i. A change of charts $\tau\colon V_i^j \supseteq U \to V \subseteq V_a^b$ thus induces a map in $\mathbf{\Psi}(\mathcal{W})$. Locally τ may therefore be represented by elements of $P_{\hat{h}}$. We conclude that P generates $\mathbf{\Psi}(\mathcal{V})$.
Since $\operatorname{dom}\mu_i^j(\lambda)$ is an open subset of V_i^j, $T\operatorname{dom}\mu_i^j(\lambda)$ may be identified with an open subset of TV_i^j. Thus a well-defined a map $\nu\colon P \to \mathbf{\Psi}(\mathcal{TV})$ is given via the assignment

$$\nu(\mu_i^j(\lambda)) := (Th^1_{t(\lambda)}|_{\nu_{\hat{h}}(\lambda)Th^1_{s(\lambda)}(T\operatorname{dom}\mu_i^j(\lambda))})^{-1}\nu_{\hat{h}}(\lambda)Th^1_{s(\lambda)}|_{T\operatorname{dom}\mu_i^j(\lambda)}.$$

Consider the map $F := \coprod_{i\in I, j\in I_j} f_i^j$. A computation yields

$$F\mu_i^j(\lambda) = \nu(\mu_i^j(\lambda))F|_{\operatorname{dom}\mu_i^j(\lambda)}.$$

Thus ν satisfies (R4a) of Definition E.2.3. Since $\nu_{\hat{h}}$ satisfies (R4b)-(R4d) of Definition E.2.3, these properties also holds for ν and we have constructed a representative $\hat{f} := (f, \mathcal{F}, [P,\nu]) \in \operatorname{Orb}(\mathcal{V}, \mathcal{TV})$ with the required properties. □

4.2.5 Proposition *Orbisections preserve local groups.*

Proof. Consider $[\hat{f}] \in \mathfrak{X}_{\mathrm{Orb}}(Q)$ together with a representative $\hat{f} = (f, \{f_i\}_{i\in I}, P_f, \nu_f)$, such that $\{f_i\}_{i\in I}$ is a canonical family with respect to some orbifold atlas $\mathcal{V}$. Consider $x \in Q$ together with an orbifold chart (V_i, G_i, ψ_i), such that $x \in \psi_i(V_i)$. Recall $f_i \in \mathfrak{X}(V_i)$, i.e. it is a vector field on V_i. Choose $z \in V$ with $\psi_i(z) = x$. We have to prove that G_z coincides with $G_{f_i(z)}$. To this end consider $\gamma \in G_{f_i(z)}$. By definition γ acts on TV via the *derived action* $\gamma.v := T\gamma(v)$. One computes now:

$$z = \pi_{TV_i} f_i(z) = \pi_{TV_i}(\gamma.f_i(z)) = \pi_{TV_i} T\gamma(f_i(z)) = \gamma.\pi_{TV_i} f_i(z) = \gamma.z$$

Thus every $\gamma \in G_{f_i(z)}$ is an element of G_z. Hence $\theta\colon G_{f_i(z)} \to G_z, \gamma \mapsto \gamma$ is an injective group homomorphism. We claim that θ is surjective. To prove this, consider $\delta \in G_z$. Observe that every $\delta \in G_z$ is a change of charts (even an open embedding of orbifold charts) and there is $g \in P$ together with an open (connected) neighborhood $\Omega_z \subseteq V$ of z, such that $\delta_{|\Omega_z} = g_{|\Omega_z}$ holds. The map $\nu_f(g)$ is a change of orbifold charts of TV into itself. Restricting to the open connected component C of $\operatorname{dom}\nu_f(g)$ which contains $f_i(z)$, [48, Lemma 2.11] implies that there is a unique $\gamma \in G$, such that $\nu_f(g)_{|C} = \gamma_{|C}$. On the open set $\Omega_z \cap f_i^{-1}(C)$ the identity

$$f_i \circ \delta_{|\Omega_z \cap f_i^{-1}(C)} = \nu_f(g) f_i|_{\Omega_z \cap f_i^{-1}(C)} = \gamma.f_i|_{\Omega_z \cap f_i^{-1}(C)} \tag{4.2.2}$$

holds. The set $\Omega_z \cap f_i^{-1}(C)$ is a non-empty open set and by Newmans theorem B.2.1 there is a non-singular $y \in \Omega_z \cap f_i^{-1}(C)$. Specializing to y, equation (4.2.2) yields:

$$f_i(\delta.y) = \gamma.f_i(y) = T\gamma f_i(y) \quad \Rightarrow \quad \delta.y = \pi_{TV_i} f_i(\delta.y) = \pi_{TV_i} T\gamma f_i(y) = \gamma.y$$

Then $\delta^{-1}\gamma.y = y$ and y being non singular forces $\gamma = \delta$. Applying this to (4.2.2) we obtain:

$$f_i(z) = f_i(\delta.z) = T\delta.f_i(z) = \delta.f_i(z)$$

In other words δ fixes $f_i(z)$ and thus δ is an element of the isotropy subgroup $G_{f_i(z)}$. Thus θ is surjective. We conclude that $\theta\colon G_z \to G_{f_i(z)}, \gamma \mapsto \gamma$ is an isomorphism of groups and the local groups Γ_z and $\Gamma_{f(z)}$ isomorphic. □

4.2.6 Proposition *Let $[\hat{f}]$ be an orbisection and $\mathcal{V} \subseteq \mathcal{U}$ an orbifold atlas. Furthermore let $\hat{f} = (f, \{ f_i \}_{i \in I}, [P_f, \nu_f]) \in \mathrm{Orb}(\mathcal{V}, \mathcal{TV})$ be a representative of $[\hat{f}]$ such that $\{ f_i \}_{i\in I}$ is a family of canonical lifts. For each element ϕ of the set of change of charts $\mathcal{C}h_\mathcal{V}$ of $\mathcal{V}$ (cf. Notation E.2.4) with $\operatorname{dom}\phi \subseteq V_i$ and $\operatorname{cod}\phi \subseteq V_j$, $(V_\alpha, G_\alpha, \psi_\alpha) \in \mathcal{V}$, $\alpha = i, j$, the identity*

$$f_j \phi = T\phi f_i|_{\operatorname{dom}\phi} \tag{4.2.3}$$

holds. The pair $(\mathcal{C}h_\mathcal{V}, \nu)$, where $\mathcal{C}h_\mathcal{V}$ is the quasi-pseudogroup of all change of charts with

$$\nu \colon \mathcal{C}h_\mathcal{V} \to \mathbf{\Psi}(\mathcal{TV}), \phi \mapsto T\phi,$$

is a representative of $[P_f, \nu_f]$.

Proof. Pick an arbitrary change of charts morphism ϕ as above and choose a representative (P_f, ν_f) of $[P_f, \nu_f]$. It suffices to prove the identity (4.2.3) on small neighborhoods of arbitrary points in $\operatorname{dom}\phi$. Let $x_0 \in \operatorname{dom}\phi$ be such a point. Since P_f generates $\mathbf{\Psi}(\mathcal{V})$ there is an open x_0 neighborhood $U_{x_0} \subseteq \operatorname{dom}\phi \subseteq V_i$ together with $\gamma^\phi_{x_0} \in P_f$, such that $\gamma_{x_0|U_{x_0}} = \phi_{|U_{x_0}}$ holds. By definition we obtain a local lift of f:

$$f_j \phi_{|U_{x_0}} = f_j \gamma^\phi_{x_0|U_{x_0}} = \nu_f(\gamma^\phi_{x_0}) f_{i|U_{x_0}}. \tag{4.2.4}$$

On the other hand, $T\phi f_{i|U_{x_0}}$ is a well-defined map, since $f_i|_{U_{x_0}} \in \mathfrak{X}(U_{x_0})$. By Lemma 4.1.4 (a), $T\phi$ is a change of charts morphism of $\mathcal{TV}$ and thus $T\phi f_i|_{U_{x_0}}$ is a local lift of f on U_{x_0}. For every $y \in U_{x_0}$ we obtain

$$T\psi_j \nu_f(\gamma^\phi_{x_0}) f_i(y) = T\psi_j T\phi f_i(y)$$

Thus there is a unique group element $g_y \in G_j$, such that $g_y.\nu_f(\gamma^\phi_{x_0}) f_i(y) = T\phi f_i(y)$ holds. In Proposition 4.2.5 we have seen that orbisections preserve local groups, whence they preserve non-singular points. Therefore lifts of orbisections map non-singular points to non-singular points. The set U_{x_0} is a non-empty open subset of V_i and by Newmans theorem B.2.1, the non-singular points of the G_i-action on V_i are dense in U_{x_0}. Using (4.2.4) for non-singular $y \in U_{x_0}$ we obtain the identities

$$\begin{aligned} &T\phi f_i(y) = g_y.\nu_f(\gamma^\phi_{x_0}) f_i(y) = g_y.f_j\phi(y) = Tg_y(f_j\phi(y)) \\ \Rightarrow \quad &\phi(y) = \pi_{TV_j} T\phi f_i(y) = \pi_{TV_i} Tg_y(f_j\phi(y)) = g_y.\phi(y). \end{aligned}$$

As change of chart maps preserve non-singular points and y is non-singluar, $g_y = \mathrm{id}_{V_j}$ follows. The maps $\nu_f(\gamma^\phi_{x_0}) f_i$ and $T\phi f_i$ therefore coincide on the non-singular points of U_{x_0}. The non-singular points are a dense in U_{x_0} and both continuous mas coincide on this set, whence $T\phi f_{i|U_{x_0}} = \nu_f(\gamma^\phi_{x_0}) f_{i|U_{x_0}}$ holds.
The quasi-pseudogroup $\mathcal{C}h_\mathcal{V}$ generates $\mathbf{\Psi}(\mathcal{V})$ and our previous considerations show that ν (as defined above) satisfies property (R4a) of Definition E.2.3. The functoriality of T implies properties (R4b)-(R4d) of Definition E.2.3 for $(\mathcal{C}h_\mathcal{V}, \nu)$. Notice that we did not change the family of lifts $\{ f_i \}_{i\in I}$. Thus $\hat{h} := (f, \{ f_i \}_{i\in I}, [\mathcal{C}h, \nu] \in \mathrm{Orb}(\mathcal{V}, \mathcal{TV})$ is a charted map, such that $[\hat{f}] = [\hat{h}]$. □

4.2.7 Remark Let M, N be smooth manifolds and $f \colon M \to N$ be a smooth map. Recall that $\sigma \in \mathfrak{X}(M)$ and $\tau \in \mathfrak{X}(N)$ are called *f-related* if $Tf \circ \sigma = \tau \circ f$ holds. Hence Proposition shows that canonical families of an orbisection are families of pairs of f-related vector fields, where f runs through the change of charts of the domains of the pair.

4.2.8 Lemma *Let $[\hat{f}]$ be an orbisection and $\mathcal{V}$ be an arbitrary representative of $\mathcal{U}$. There is a refinement $\mathcal{V}'$ of $\mathcal{V}$ and a representative $\hat{h} := (f, \{ h_i \}_{i \in I}, [P, \nu]) \in \operatorname{Orb}(\mathcal{V}', \mathcal{T}\mathcal{V}')$ of $[\hat{f}]$, such that $\{ h_i \}_{i \in I}$ is a family of canonical lifts for $[\hat{f}]$.*

Proof. By Lemma 4.2.4 we may choose a representative $\mathcal{W}$ of $\mathcal{U}$ indexed by I and a representative $\hat{g} = (f, \{ g_i \}_{i \in I}, [P, \nu]) \in \operatorname{Orb}(\mathcal{W}, \mathcal{T}\mathcal{W})$ of $[\hat{f}]$, such that $\{ g_i \}_{i \in I}$ is a canonical family. Choose a common refinement $\mathcal{V}'$ of $\mathcal{W}$ and $\mathcal{V}$. The refinement $\mathcal{V}'$ induces a common refinement $\mathcal{T}\mathcal{V}'$ of $\mathcal{T}\mathcal{V}$ and $\mathcal{T}\mathcal{W}$, since open embeddings of orbifold maps are mapped to open embeddings of orbifold charts via the tangential functor T. Let $\mathcal{V}'$ be indexed by J and $\alpha \colon J \to I$ be the map, which assigns to $j \in J$ the codomain of the open embedding of orbifold charts $\lambda_j \colon (V', G', \pi') \to (W_{\alpha(j)}, H_{\alpha(j)}, \psi_{\alpha(j)})$. The family $\{ g_i \}_{i \in I}$ is a canonical family, therefore

$$g_{\alpha(j)} \lambda_j (V_j') = g_{\alpha(j)} (\operatorname{Im} \lambda_j) \subseteq T \operatorname{Im} \lambda_j$$

Define the maps $h_j := (T\lambda_j)^{-1} g_{\alpha(j)} \lambda_j \colon V_j \to TV_j$. Then Lemma E.4.2 assures that there is a pair (P, ν), such that $\hat{h} := (f, \{ h_j \}_{j \in J}, [P, \nu])$ is a representative of $[\hat{f}]$. A computation yields

$$\pi_{TV_j} h_j = \pi_{TV_j} (T\lambda_j)^{-1} g_{\alpha(j)} \lambda_j = \lambda_j^{-1} \pi_{TW_{\alpha(j)}} g_{\alpha(j)} \lambda_j = \operatorname{id}_{V_j}$$

for each $j \in J$. In conclusion $\{ h_j \}_{j \in J}$ is a canonical family and the domain atlas of $\hat{h}$ is a refinement of $\mathcal{V}$. □

The results obtained so far show that each orbisection possesses representatives, whose lifts form canonical families for suitable refinements of $\mathcal{U}$. We will now prove a converse: For each orbisection and an arbitrary orbifold atlas, there is a representative, whose lifts form a canonical family with respect to the given atlas. This result is quite surprising since in general maps of orbifolds need not have lifts on an orbifold chart chosen in advance.

4.2.9 Proposition *Let $[\hat{f}] \in \mathfrak{X}_{Orb}(Q)$ and $\mathcal{W}$ be an arbitrary representative of $\mathcal{U}$ indexed by J. There exists a representative $\hat{g} = (f, \{ g_i \}_{i \in I}, [P, \nu]) \in \operatorname{Orb}(\mathcal{W}, \mathcal{T}\mathcal{W})$, such that $\{ g_i \}_{i \in I}$ is a canonical family with respect to $\mathcal{W}$.*

Proof. Lemma 4.2.8 allows us to choose a refinement $\mathcal{V}$ of $\mathcal{W}$ indexed by I and a representative $\hat{h} := (f, \{ f_{V_i} \}_{i \in I}, [P, \nu]) \in \operatorname{Orb}(\mathcal{V}, \mathcal{T}\mathcal{V})$ of $[\hat{f}]$, such that $\{ f_{V_i} \}_{i \in I}$ is a family of canonical lifts for $[\hat{f}]$. Let $(W_j, G_j, \psi_j) \in \mathcal{W}$ be an arbitrary orbifold chart. We have to construct a local lift of f on (W_j, G_j, ψ_j). This will be achieved by an application of Zorns Lemma (cf. [20, I.4]): Recall the notation introduced in Lemma 4.1.4 2.:

$$\mathcal{C}h_{V_i, W_j} := \{ \lambda \colon V_i \supseteq \operatorname{dom} \lambda \to \operatorname{cod} \lambda \subset W_j | \lambda \text{ is a change of charts} \}.$$

Consider the set $\operatorname{Loc}(W_j)$ of all smooth maps $h \colon W_j \supseteq U \to TU \subseteq TW_j$ on arbitrary open and connected sets $U \subseteq W_j$, such that $T\psi_j h = f \psi_{j|\operatorname{dom} h}$.

$$\mathbb{L}(W_j) := \left\{ h \in \operatorname{Loc}(W_j) \middle| \forall (V_i, G_i, \pi_i) \in \mathcal{V} \text{ and } \forall \lambda \in \mathcal{C}h_{V_i, W_j},\ h_{|\operatorname{cod} \lambda \cap \operatorname{dom} h} = T\lambda f_{V_i} \lambda^{-1}_{|\operatorname{cod} \lambda \cap \operatorname{dom} h} \right\}$$

In order to use Zorns Lemma, we have to verify various facts: First of all $\mathbb{L}(W_j)$ is non-empty. To see this, simply choose $(V_i, G_i, \pi_i) \in \mathcal{V}$, such that $\pi_i(V_i) \cap \psi_j(W_j) \neq \emptyset$. Pick $x \in V_i$, with $\pi_i(x) \in \psi_j(W_j)$ and observe that by compatibility of charts, there is a change of charts map $\lambda_x \colon V_i \supseteq \Omega_x^i \to \Omega_x^j \subseteq W_j$. The lift $f_{V_i} \colon V_i \to TV_i$ induces a lift on Ω_x^j via

$$f_{\Omega_x^j} \colon \Omega_x^j \to T\Omega_x^j, f_{\Omega_x^j} := T\lambda_x f_{V_i} \lambda_x^{-1}.$$

This map is smooth and Ω_x^j is a connected open set. A computation shows $T\psi_j f_{\Omega_x^j} = f\psi_j|_{\Omega_x^j}$ and thus $f_{\Omega_x^j} \in \mathrm{Loc}(W_j)$. Considering $\mu \in \mathcal{C}h_{V_k,W_j}$, $k \in I$, we have to check that $f_{\Omega_x^j}$ satisfies the defining identity of $\mathbb{L}(W_j)$. As Ω_x^j is open, on every connected component C of $\operatorname{cod}\mu \cap \Omega_x^j$, we obtain a change of charts $\lambda_x^{-1}\mu_{|\mu^{-1}(C)} \in \mathcal{C}h_{V_k,V_i}$. Without loss of generality $P = \mathcal{C}h_{\mathcal{V}}$ holds and ν is the map constructed in Proposition 4.2.6. A computation then leads to:

$$T\mu f_{V_j} \mu_{|C}^{-1} \overset{(4.2.3)}{=} T\mu (T(\lambda_x^{-1}\mu))^{-1} f_{V_i} (\lambda_x^{-1}\mu)\mu_{|C}^{-1} = T\lambda_x f_{V_i} \lambda_{x|C}^{-1} = f_{\Omega_x^j|C}.$$

Since μ was arbitrary, $f_{\Omega_x^j}$ is an element of $\mathbb{L}(W_j)$. The set $\mathbb{L}(W_j)$ is thus non-empty and the construction shows that for $y \in W_j$, there is an element $g_y \in \mathbb{L}(W_j)$, such that $y \in \operatorname{dom} g_y$. Define a partial ordering "$\preceq$" on $\mathbb{L}(W_j)$ via

$$g \preceq f :\Leftrightarrow \operatorname{dom} g \subseteq \operatorname{dom} f \text{ and } f_{|\operatorname{dom} g} = g.$$

We have to prove that $(\mathbb{L}(W_j), \preceq)$ is inductively ordered. Let $\mathcal{S}$ be an arbitrary chain in $\mathbb{L}(W_j)$. If $\mathcal{S}$ is the empty chain, every element of $\mathbb{L}(W_j)$ is an upper bound of $\mathcal{S}$. If $\mathcal{S}$ is non-empty, e.g. $\mathcal{S} = \{ g_r \}_{r \in R}$ define

$$\operatorname{dom} G_{\mathcal{S}} := \bigcup_{r \in R} \operatorname{dom} g_r \text{ and } G_{\mathcal{S}} \colon \operatorname{dom} G_{\mathcal{S}} \to T \operatorname{dom} G_{\mathcal{S}},\ G_{\mathcal{S}} := \bigcup_{r \in R} g_r.$$

The set $\operatorname{dom} G_{\mathcal{S}}$ is open. Since [20, Corollary 6.1.10] implies that the union of an ascending chain of connected sets is again connected, $\operatorname{dom} G_{\mathcal{S}}$ is connected. The map $G_{\mathcal{S}}$ is smooth and satisfies $T\psi_j G_{\mathcal{S}} = f\psi_{j|\operatorname{dom} G_{\mathcal{S}}}$, since every g_r is a map with these properties. Consider a change of charts $\lambda \in \mathcal{C}h_{V_k,W_j}$ and the set $A := \operatorname{cod}\lambda \cap \operatorname{dom} G_{\mathcal{S}}$. This set is covered by the domains $\operatorname{dom} g_r$ and on each domain the map $G_{\mathcal{S}}$ coincides with some g_r. As each $g_r \in \mathbb{L}(W_j)$, $G_{\mathcal{S}}$ satisfies $G_{\mathcal{S}|A} = T\lambda f_{V_k} \lambda_{|A}^{-1}$, whence $G_{\mathcal{S}}$ is in $\mathbb{L}(W_j)$. By construction $G_{\mathcal{S}}$ is an upper bound of $\mathcal{S}$ and thus $(\mathbb{L}(W_j), \preceq)$ is inductively ordered. Zorns Lemma shows that there is a maximal element $f_{W_j} \in \mathbb{L}(W_j)$.
Arguing indirectly we assume that $\operatorname{dom} f_{W_j} \neq W_j$. Hence there is $z \in \partial \operatorname{dom} f_{W_j}$, where $\partial \operatorname{dom} f_{W_j}$ is the boundary of the open connected set $\operatorname{dom} f_{W_j}$ in the connected manifold W_j. There is an element $f_z \in \mathbb{L}(W_j)$, such that $z \in \operatorname{dom} f_z$. Without loss of generality, we may assume that $f_z = T\lambda f_{V_k} \lambda_{|\operatorname{dom} f_z}^{-1}$ holds, for some $(V_k, G_k, \pi_k) \in \mathcal{V}$ and a lift $f_{V_k} \in \{ f_{V_i} \}_{i \in I}$. Since $\operatorname{dom} f_z$ is open and $z \in \partial \operatorname{dom} f_{W_j}$, $\operatorname{dom} f_z \cap \operatorname{dom} f_{W_j} \neq \emptyset$. Again by [20, Corollary 6.1.10] $\operatorname{dom} f_{W_j} \cup \operatorname{dom} f_z$ is open, connected and properly contains $\operatorname{dom} f_{W_j}$. On $\operatorname{dom} f_{W_j} \cap \operatorname{dom} f_z$ the definition of f_z and $f_{W_j} \in \mathbb{L}(W_j)$ yield:

$$f_z|_{\operatorname{dom} f_{W_j} \cap \operatorname{dom} f_z} = T\lambda f_{V_k} \lambda^{-1}|_{\operatorname{dom} f_{W_j} \cap \operatorname{dom} f_z} = T\lambda f_{V_k} \lambda^{-1}|_{\operatorname{cod}\lambda \cap \operatorname{dom} f_{W_j}} = f_{W_j}|_{\operatorname{dom} f_{W_j} \cap \operatorname{dom} f_z}.$$

In particular $f_z \cup f_{W_j} \colon \operatorname{dom} f_z \cup \operatorname{dom} f_{W_j} \to T(\operatorname{dom} f_z \cup \operatorname{dom} f_{W_j})$ is a well-defined smooth map, with $T\psi_j f_z \cup f_{W_j} = f\psi_{j|\operatorname{dom} f_z \cup f_{W_j}}$ on the open connected set $\operatorname{dom} f_{W_j} \cup \operatorname{dom} f_z$. An analogous

argument as in the construction of the upper bound for a chain in $\mathbb{L}(W_j)$ shows $f_z \cup f_{W_j} \in \mathbb{L}(W_j)$. This contradicts the maximality of f_{W_j}, whence $\operatorname{dom} f_{W_j} = W_j$ holds.
Observe that f_{W_j} is a vector field, since each f_{V_i} is a vector field and $f_{W_j} \in \mathbb{L}(W_j)$. Repeating the construction for every chart in $\mathcal{W}$, we obtain a lift of f in $\mathfrak{X}(W_j)$ for each $(W_j, H_j, \psi_j) \in \mathcal{W}$.
Set $\nu_{\mathcal{V}} \colon \mathcal{C}h_{\mathcal{V}} \to \mathbf{\Psi}(\mathcal{T}\mathcal{W}), h \mapsto Th$. We have to show that the pair $(\mathcal{C}h_{\mathcal{V}}, \nu_{\mathcal{V}})$ together with the family $\{ f_{W_j} \}_{\mathcal{W}}$ satisfies property (R4a) of Definition E.2.3. To this end, pick an arbitrary change of charts $h \in \mathcal{C}h_{\mathcal{V}}$ and let $\operatorname{dom} h \subseteq W_r$, $\operatorname{cod} h \subseteq W_s$ for some charts $(W_\alpha, G_\alpha, \pi_\alpha) \in \mathcal{W}$, $\alpha = r, s$. It suffices to prove the identity $f_{W_s} h = T h f_{W_r}$ on an open covering of $\operatorname{dom} h$. Let $x \in \operatorname{dom} h$ and choose a chart $(V_i, G_i, \pi_i) \in \mathcal{V}$ and a change of charts $\lambda \colon V_i \supseteq U \to \Omega \subseteq \operatorname{dom} h$, with $x \in \Omega$. By construction $f_{W_r}|_\Omega = T\lambda f_{V_i} \lambda^{-1}|_\Omega$ holds on Ω and since $h\lambda \in \mathcal{C}h_{V_i, W_s}$, the construction of f_{W_s} implies

$$\begin{aligned} \nu_{\mathcal{V}}(h) f_{W_r}|_\Omega = T h f_{W_r}|_\Omega = T h T \lambda f_{V_i} \lambda^{-1}|_\Omega &= T(h\lambda) f_{V_i} \lambda^{-1} h^{-1} h|_\Omega \\ &= T(h\lambda) f_{V_i} (h\lambda)^{-1} h|_\Omega = f_{W_s} h|_\Omega. \end{aligned}$$

This proves property (R4a) for the pair $(\mathcal{C}h_{\mathcal{V}}, \nu_{\mathcal{V}})$ and the family of lifts $\{ f_{W_j} | (W_j, H_j, \pi_j) \in \mathcal{W} \}$. By Proposition 4.2.6 we obtain a map $\hat{g} := (f, \{ f_{W_j} | (W_j, G_j, \psi_j) \in \mathcal{W} \}, [\mathcal{C}h_{\mathcal{V}}, \nu_{\mathcal{V}}]) \in \operatorname{Orb}(\mathcal{W}, \mathcal{T}\mathcal{W})$ and the family $\{ f_{W_j} \}_{\mathcal{W}}$ is a canonical family for $[\hat{g}]$.
The atlas $\mathcal{V}$ is a refinement of $\mathcal{W}$, thus for every $i \in I$, there is an embedding of orbifold charts $\lambda_i \colon (V_i, G_i, \pi_i) \to (W_{\alpha(i)}, G_{\alpha(i)}, \psi_{\alpha(i)})$. By construction we obtain $f_i = T\lambda^{-1} f_{W_{\alpha(i)}} \lambda_i$ and therefore every lift f_i is induced by a suitable lift of $\hat{g}$. Following Definition E.4.3 we have $\hat{g} \sim \hat{h}$ and the classes $[\hat{g}]$ and $[\hat{f}]$ coincide. Thus the lifts are a canonical family of $[\hat{f}]$ with respect to $\mathcal{W}$. $\square$

Proposition 4.2.9 shows that every orbisection may be identified in every given atlas with a unique family of canonical representatives. In particular, orbisections satisfy analogous properties as C^∞-sections into the tangent bundle in the sense of [14, After Remark 4.1.8].

4.2.10 Remark

(a) A family $\mathcal{F}$ of vector fields on an orbifold atlas $\mathcal{V}$, which satisfies equation (4.2.3), induces a continuous map $F \colon Q \to \mathcal{T}Q$ (cf. the proof of Proposition 4.3.1 for the explicit construction) such that
- $(F, \mathcal{F}, [\mathcal{C}h_{\mathcal{V}}, \nu]) \in \operatorname{Orb}(\mathcal{V}, \mathcal{T}\mathcal{V})$ with $\nu \colon \mathcal{C}h_{\mathcal{V}} \to \mathbf{\Psi}(\mathcal{T}\mathcal{V}), \lambda \mapsto T\lambda$,
- $\mathcal{F}$ is a canonical family .

Vice versa, if $(f, \{ f_i | i \in I \}, [P_f, \nu_f])$ is a representative of an orbisection, whose lifts form a canonical family with respect to an atlas $\mathcal{V}$, the above construction for $\{ f_i | i \in I \}$ yields the map f. Lemma 4.2.3 implies that an orbisection is uniquely determined by its family of canonical lifts with respect to any atlas $\mathcal{V}$. This induces a one to one correspondence between the set of orbisections and families of vector fields for some orbifold atlas $\mathcal{V}$, which satisfy (4.2.3).

(b) Notice that (a) implies: For $[\hat{f}] \in \mathfrak{X}_{\mathrm{Orb}}(Q)$ and $(U, G, \psi) \in \mathcal{U}$ there is a unique vector field $\hat{f}_U \in \mathfrak{X}(U)$, such that for $\hat{f} = (f, \{ f_i | i \in I \}, [P, \nu])$ the identity $T\psi \hat{f}_U = f\psi$ holds.

(c) The canonical lift of the zero orbisection with respect to some orbifold chart (U, G, ψ) is the zero-section in $\mathfrak{X}(U)$. If $[\hat{f}] \in \mathfrak{X}_{\mathrm{Orb}}(Q)$ is an orbisection and $(U, G, \psi) \in \mathcal{U}$ is some chart, such that $\psi(U) \cap \operatorname{supp}[\hat{f}] = \emptyset$, then the canonical lift of $[\hat{f}]$ on U is the zero section in $\mathfrak{X}(U)$.

4.3. Spaces of Orbisections

In this section spaces of orbisections are studied. We shall endow them with the structure of a real topological vector space:

4.3.1 Proposition *The set $\mathfrak{X}_{Orb}(Q)$ of orbisections is a real vector space with pointwise vector space operations. The zero element $\mathbf{0}_{Orb} \in \mathfrak{X}_{Orb}(Q)$ of $\mathfrak{X}_{Orb}(Q)$ is called the* zero orbisection. *Endowing $\mathfrak{X}_{Orb}(Q)$ with this vector space structure, the sets $\mathfrak{X}_{Orb}(Q)_K \subseteq \mathfrak{X}_{Orb}(Q)_c \subseteq \mathfrak{X}_{Orb}(Q)$ become linear subspaces.*

Proof. Let $[\hat{f}], [\hat{g}] \in \mathfrak{X}_{\mathrm{Orb}}(Q)$ and choose an arbitrary representative $\mathcal{V}$ of the maximal orbifold atlas $\mathcal{U}$, indexed by some set I. By Proposition 4.2.9 we may choose unque representatives of orbifold maps $\hat{f} = (f, \{ f_i | i \in I \}, P_f, \nu_f) \in \mathrm{Orb}(\mathcal{V}, \mathcal{T}\mathcal{V})$ of $[\hat{f}]$ and $\hat{g} = (g, \{ g_i | i \in I \}, P_g, \nu_g) \in \mathrm{Orb}(\mathcal{V}, \mathcal{T}\mathcal{V})$ of $[\hat{g}]$, such that the families of lifts are canonical families. Without loss of generality $P_f = P_g = \mathcal{C}h_{\mathcal{V}}$ and $\nu_f(\lambda) = \nu_g(\lambda) = T\lambda$ hold by Proposition 4.2.6. By construction for each $i \in I$ the lifts are vector fields $f_i, g_i \in \mathfrak{X}(V_i)$. Recall from [15, 2.7] that the vector space structure on $\mathfrak{X}(V_i)$ is induced by pointwise operations. We define the vector space operations on $\mathfrak{X}_{\mathrm{Orb}}(Q)$ via the following construction:

For $z \in \mathbb{R}$ consider $f_i + zg_i \colon V_i \to TV_i \in \mathfrak{X}(V_i)$. Remember that tangent maps act as linear maps on each tangent space. For every change of charts $\lambda \in \mathbf{\Psi}(\mathcal{V})$ with $\operatorname{dom}\lambda \subseteq V_i$ and $\operatorname{cod}\lambda \subseteq V_j$ we obtain:

$$\begin{aligned}(f_j + zg_j)\lambda(p) &= f_j(\lambda(p)) + zg_j(\lambda(p)) = \nu_f(\lambda)f_i(p) + z\nu_g(\lambda)g_i(p) \\ &= T_p\lambda(f_i(p)) + zT_p\lambda(zg_i(p)) = T_p\lambda(f_i(p) + zg_i(p)) \\ &=: \nu_{f+zg}(\lambda)(f_i(p) + zg_i(p))\end{aligned} \tag{4.3.1}$$

Define the quasi-pseudogroup $P_{f+zg} := \mathcal{C}h_{\mathcal{V}}$ together with $\nu_{f+zg} \colon P_{f+zg} \to \mathbf{\Psi}(\mathcal{T}\mathcal{V}),\ \lambda \mapsto T\lambda$. The pair (P_{f+zg}, ν_{f+zg}) and the family $\{ f_i + zg_i | i \in I \}$ satisfy properties (R4a)-(R4d) of Definition E.2.3. We have to provide a continuous map $f + zg \colon Q \to \mathcal{T}Q$, such that every $f_i + zg_i$ is a local lift for this map.

Observe that for $(V_i, G_i, \psi_i) \in \mathcal{V}$ the action of G_i on TV_i is the derived action. Thus $\gamma \in G_i$ acts via the linear map $T_p\gamma$ on the tangent space T_pV_i. Specializing (4.3.1) one obtains $(f_i + zg_i)(\gamma.x) = T\gamma.(f_i(x) + zg_i(x))$. Define for each chart $(V_i, G_i, \psi_i) \in \mathcal{V}$ a map on $\psi_i(V_i)$ by:

$$f + zg|_{\psi_i(V_i)} \colon \psi(V_i) \to T\psi_i(TV_i), x \mapsto T\psi_i \circ (f_i + zg_i)\psi^{-1}(x)$$

This map is well-defined, since for every choice in $\psi^{-1}(x)$ the assignment yields the same image. To see this choose $p, p' \in \psi^{-1}(x)$, there is some $\gamma_i \in G_i$ such that $\gamma_i.p = p'$ holds. As $T\gamma_i$ is a change of charts, we deduce $T\psi_i(f_i + zg_i)(p') = T\psi_i(f_i + zg_i)(\gamma.p) = T\psi_i T\gamma(f_i + zg_i)(p) = T\psi_i(f_i + zg_i)(p)$. The maps $T\psi_i, (f_i + zg_i)$ are continuous and ψ_i is open as a composition of the quotient map to an orbit space with a homeomorphism. Thus $f + zg|_{\psi_i(V_i)}$ is continuous and $f_i + zg_i$ is a lift for this map. We claim that for every pair $(i,j) \in I \times I$ the maps $f + zg|_{\psi_i(V_i)}$ and $f + zg|_{\psi_j(V_j)}$ coincide on $\psi_j(V_j) \cap \psi_i(V_i)$. If this were true, then $f + zg \colon Q \to \mathcal{T}Q, x \mapsto f + zg|_{\psi_i(V_i)}(x),\ \forall x \in \psi_i(V_i)$ is a well-defined continuous map. We obtain a charted orbifold map

$$\widehat{f + zg} := (f + zg, \{ f_i + zg_i | i \in I \}, P_{f+zg}, \nu_{f+zg}) \in \mathrm{Orb}(\mathcal{V}, \mathcal{T}\mathcal{V}),$$

such that each lift $f_i + zg_i$ is a vector field. Hence $\{ f_i + zg_i | i \in I \}$ is a canonical family with respect to the atlas $\mathcal{V}$ and $[\widehat{f + zg}] \in \mathfrak{X}_{\mathrm{Orb}}(Q)$ holds. Proof of the claim: Consider $x \in \psi_i(V_i) \cap \psi_j(V_j)$. For every pair $y_\alpha \in \psi_\alpha^{-1}(x)$, $\alpha \in \{ i, j \}$, there is a change of charts morphism $\lambda \colon V_i \supseteq U \to V \subseteq V_j$, such that $\lambda(y_i) = y_j$. Again by (4.3.1), the claim follows as

$$\begin{aligned} f + zg|_{\psi_j(V_j)}(x) &= T\psi_j(f_j + zg_j)(y_j) = T\psi_j(f_j + tg_j)(\lambda(y_i)) \\ &= T\psi_j T\lambda(f_i + zg_i)(y_i) = T\psi_i(f_i + zg_i)(p) = f + zg|_{\psi_i(V_i)}(x) \end{aligned}$$

It remains to show that the construction does not depend on the atlas $\mathcal{V}$. Let $\mathcal{V}'$ be another representative of $\mathcal{U}$ and $\hat{f}'$ resp. $\hat{g}'$ be representatives of $[\hat{f}]$ resp. $[\hat{g}]$, whose families of lifts form canonical families with respect to $\mathcal{V}'$. By Lemma 2.6.2, we may choose a common refinement of $\mathcal{V}$ and $\mathcal{V}'$. The definition of equivalence of orbifold maps implies that the classes will be equal if the induced lifts on this refinement coincide. Without loss of generality we may assume that $\mathcal{V}'$ refines $\mathcal{V}$: Let $\mathcal{V}' = \{ (W_k, H_k, \phi_k) | k \in K \}$ and $\alpha \colon K \to J$ be the map, which assigns to $k \in K$ an element of I, such that there is an embedding of orbifold charts $\lambda_k \colon (W_k, H_k, \phi_k) \to (V_{\alpha(k)}, G_{\alpha(k)}, \psi_{\alpha(k)})$. The atlas $\mathcal{TV}'$ for $\mathcal{T}Q$ is a refinement of $\mathcal{TV}$. In particular $T\lambda_k$ is an embedding of $(TW_k, H_k, T\phi_k)$ into $(TV_{\alpha(k)}, H_{\alpha(k)}, T\psi_{\alpha(k)})$. Let $\hat{f}' = (f, \{ f'_k | k \in K \}, P'_f, \nu'_f)$ and $\hat{g}' = (g, \{ g'_k | k \in K \}, P'_g, \nu'_g)$. Withoput loss of generality $P'_f = P'_g = \mathcal{C}h_{\mathcal{W}}$ and $\nu'_f(\lambda) = \nu'_g(\lambda) = T\lambda$, $\forall \lambda \in \mathcal{C}h_{\mathcal{W}}$ hold. Since $\{ f_i | i \in I \}$ and $\{ g_i | i \in I \}$ are families of vector fields, we obtain induced vector fields $T\lambda_k^{-1} f_{\alpha(k)} \lambda_k$ and $T\lambda_k^{-1} g_{\alpha(k)} \lambda_k$. These induced lifts are canonical lifts for $[\hat{f}]$ resp. $[\hat{g}]$ with respect to $\mathcal{W}$ by Lemma 4.2.8. As canonical lifts are unique by Lemma 4.2.3, we obtain

$$f'_k = T\lambda_k^{-1} f_{\alpha(k)} \lambda_k, \quad g'_k = T\lambda_k^{-1} g_{\alpha(k)} \lambda_k.$$

Constructing $\widehat{f' + zg'} \in \mathrm{Orb}(\mathcal{V}', \mathcal{TV}')$ as above, the lifts of this map satisfy:

$$T\lambda_k^{-1}(f_{\alpha(k)} + zg_{\alpha(k)})\lambda_k = T\lambda_k^{-1} f_{\alpha(k)} \lambda_k + zT\lambda_k^{-1} g_{\alpha(k)} \lambda_k = f'_k + zg'_k, \ \forall k \in K$$

From this observation readily $\widehat{f + zg} \sim \widehat{f' + zg'}$ follows. A vector space structure on $\mathfrak{X}_{\mathrm{Orb}}(Q)$ is thus defined via the assignment:

$$[\hat{f}] + z[\hat{g}] := [\widehat{f + zg}]$$

Clearly $\mathbf{0}_{\mathrm{Orb}} \in \mathfrak{X}_{\mathrm{Orb}}(Q)_K \subseteq \mathfrak{X}_{\mathrm{Orb}}(Q)_c$ holds, whence these subsets are not empty. The last claim follows from the definitions: For $[\hat{f}], [\hat{g}] \in \mathfrak{X}_{\mathrm{Orb}}(Q)_c$ with $\mathrm{supp}[\hat{f}] \subseteq K$ and $\mathrm{supp}[\hat{g}] \subseteq L$ with $K, L \subseteq Q$ compact, one obtains $\mathrm{supp}([\hat{f}] + z[\hat{g}]) \subseteq \mathrm{supp}[\hat{f}] \cup \mathrm{supp}[\hat{g}] \subseteq K \cup L$. Therefore $\mathfrak{X}_{\mathrm{Orb}}(Q)_K$ and $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$ are linear subspaces. □

Our goal for the remainder of this section is to topologize the vector spaces $\mathfrak{X}_{\mathrm{Orb}}(Q)$ and $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$. If Q is a compact topological space, then $\mathfrak{X}_{\mathrm{Orb}}(Q)$ will be a Fréchet space.

4.3.2 Lemma *Let $(Q, \mathcal{U})$ be an orbifold and $\mathcal{V}$ an arbitrary representative of $\mathcal{U}$ indexed by I. There is a bijection identifying each $[\hat{f}] \in \mathfrak{X}_{Orb}(Q)$ with a unique representative $\hat{f}_{\mathcal{V}}$, whose lifts $\left\{ \hat{f}_{U_i} \middle| (U_i, G_i, \psi_i) \in \mathcal{V} \right\}$ form a canonical family for $[\hat{f}]$ with respect to $\mathcal{V}$.*

(a) *The map*

$$\Lambda_{\mathcal{V}} \colon \mathfrak{X}_{Orb}(Q) \to \prod_{i \in I} \mathfrak{X}(U_i), \hat{f}_{\mathcal{V}} \mapsto (f_{U_i})_{i \in I}$$

is a linear injection into a direct product of topological vector spaces (cf. Section C.3 for information on $\mathfrak{X}(U)$), whose image is the closed vector subspace

$$H := \left\{ (f_i)_{i \in I} \in \prod_{i \in I} \mathfrak{X}(U_i) \middle| \forall \lambda \in \mathcal{C}h_{\mathcal{V}}, \operatorname{dom}\lambda \subseteq U_i, \operatorname{cod}\lambda \subseteq U_j,\ f^j\lambda = T\lambda f_{i|\operatorname{dom}\lambda} \right\}$$

(b) *If $\mathcal{V}$ is a locally finite atlas, the map*

$$\Lambda_{\mathcal{V}} \colon \mathfrak{X}_{Orb}(Q)_c \to \bigoplus_{i \in I} \mathfrak{X}(U_i), \hat{f}_{\mathcal{V}} \mapsto (f_{U_i})_{i \in I}$$

is a linear injection into the direct sum of topological vector spaces (cf. [37, 4.3]). Taking identifications, its image is the closed vector subspace $H \cap \bigoplus_{i \in I} \mathfrak{X}(U_i)$.

Proof. (a) For $[\hat{f}] \in \mathfrak{X}_{\mathrm{Orb}}(Q)$, we let $\left\{ \hat{f}_{U_i} \middle| i \in I \right\}$ be the family of canonical lifts with respect to $\mathcal{V}$. Proposition 4.2.6 shows that $\operatorname{Im}\Lambda_{\mathcal{V}}$ is contained in H. Remark 4.2.10 (a) implies that $\Lambda_{\mathcal{V}}$ is injective and $\operatorname{Im}\Lambda_{\mathcal{V}} = H$ holds. The vector space operations of $\mathfrak{X}_{\mathrm{Orb}}(Q)$ are defined via poitnwise operations for families of vector fields. Hence by definition, $\Lambda_{\mathcal{V}}$ is a linear map.
We have to show that H is a closed vector subspace. Consider $\lambda \in \mathcal{C}h_{\mathcal{V}}$ and $g_\lambda \colon \mathfrak{X}(\operatorname{dom}\lambda) \to \mathfrak{X}(\operatorname{cod}\lambda),\ X \mapsto T\lambda \circ X \circ \lambda^{-1}$. The map g_λ is well-defined and we claim that g_λ is continuous. To prove the claim several facts about the topology of $\mathfrak{X}(U)$ from Section C.3:
Consider the change of charts $\lambda \colon V_{s(\lambda)} \supseteq \operatorname{dom}\lambda \to \operatorname{cod}\lambda \subseteq V_{t(\lambda)}$. Its domain and codomain are open submanifolds of $V_{s(\lambda)}$ resp. $V_{t(\lambda)}$. Choose an arbitrary atlas $\{ \psi_k | k \in K \}$ of $\operatorname{cod}\lambda$ as an open submanifold of $V_{t(\lambda)}$. Since $\lambda \colon \operatorname{dom}\lambda \to \operatorname{cod}\lambda$ is a diffeomorphism, $\{ \psi_k\lambda | k \in K \}$ is an atlas of $\operatorname{dom}\lambda$. The topology on $\mathfrak{X}(\operatorname{dom}\lambda)$ resp. $\mathfrak{X}(\operatorname{cod}\lambda)$ is the initial topology with respect to the family $(\theta_{\psi_k\lambda})_{k \in K}$, resp. $(\theta_{\psi_k})_{k \in K}$ by Lemma C.3.2. Thus g_λ is continuous iff the composition $\theta_{\psi_k} \circ g_\lambda$ is continuous for each $k \in K$. The canonical lifts considered are λ-related vector fields, whence for $k \in K$ we obtain a commutative diagramm:

$$\begin{tikzcd}
\mathfrak{X}(\operatorname{dom}\lambda) \arrow[r, "g_\lambda"] \arrow[d, "\theta_{\psi_k\lambda}"'] & \mathfrak{X}(\operatorname{cod}\lambda) \arrow[d, "\theta_{\psi_k}"] \\
C^\infty(\lambda^{-1}(U_{\psi_k}), \mathbb{R}^n) \arrow[r, "C^\infty(\lambda^{-1}{,}\mathbb{R}^n)"] & C^\infty(U_{\psi_k}, \mathbb{R}^n)
\end{tikzcd}$$

The map in the lower row is a pullback by a smooth map, which is continuous by [23, Lemma 3.7]. The map $\theta_{\psi_k\lambda}$ is continuous and we conclude that

$$\theta_{\psi_k} \circ g_\lambda = C^\infty(\lambda^{-1}, \mathbb{R}^n) \circ \theta_{\psi_k\lambda}$$

is a continuous map. Since k was arbitrary g_λ is continuous as claimed. By [25, Lemma F.15 (a)][3] the map $\mathrm{res}^{U_{s(\lambda)}}_{\mathrm{dom}\,\lambda} \colon \mathfrak{X}\left(U_{s(\lambda)}\right) \to \mathfrak{X}\left(\mathrm{dom}\,\lambda\right)$, which sends vector fields to their restriction, is continuous. We obtain a continuous map $G_\lambda := g_\lambda \circ \mathrm{res}^{U_{s(\lambda)}}_{\mathrm{dom}\,\lambda} \colon \mathfrak{X}\left(U_{s(\lambda)}\right) \to \mathfrak{X}\left(\mathrm{cod}\,\lambda\right)$, whence the subspace

$$A_\lambda := \left\{ (f_i) \in \prod_{(U_i,G_i,\psi_i)\in\mathcal{A}} \mathfrak{X}\left(U_i\right) \middle| \mathrm{res}^{U_{t(\lambda)}}_{\mathrm{cod}\,\lambda}(f_{t(\lambda)}) = G_\lambda(f_{s(\lambda)}) \right\}$$

is closed. The space H is the intersection $H = \bigcap_{\lambda\in\mathcal{C}h_{\mathcal{A}}} A_\lambda$ and thus H is a closed subspace.

(b) The atlas $\mathcal{V}$ is locally finite and thus only finitely many charts intersect a given compact set. In particular $\Lambda_{\mathcal{V}}$ is well defined. The canonical injection $I \colon \bigoplus_{i\in I}\mathfrak{X}\left(U_i\right) \to \prod_{i\in I}\mathfrak{X}\left(U_i\right)$ is continuous by [37, 4.3.1] and $I^{-1}(H) = H \cap \bigoplus_{i\in I}\mathfrak{X}\left(U_i\right)$ is a closed subset of $\bigoplus_{i\in I}\mathfrak{X}\left(U_i\right)$. Again by Proposition 4.2.6 $\mathrm{Im}\,\Lambda_{\mathcal{V}}$ is contained in $I^{-1}(H)$ and by remark 4.2.10 $\Lambda_{\mathcal{V}}$ is injective and $\mathrm{Im}\,\Lambda_{\mathcal{V}} = I^{-1}(H) = H \cap \bigoplus_{i\in I}\mathfrak{X}\left(U_i\right)$.

□

4.3.3 Definition

(a) As usual $\mathcal{U}$ is the maximal atlas of the orbifold $(Q,\mathcal{U})$. Endow $\mathfrak{X}_{\mathrm{Orb}}\left(Q\right)$ with the locally convex vector topology making the linear map

$$\Lambda \colon \mathfrak{X}_{\mathrm{Orb}}\left(Q\right) \to \prod_{(U,G,\psi)\in\mathcal{U}} \mathfrak{X}\left(U\right), \ [\hat{f}] \mapsto (f_U)_{(U,G,\psi)\in\mathcal{U}}$$

a topological embedding. Here we have used the unique lifts f_U constructed in remark 4.2.10. We call this topology the *orbisection topology* and note that it is the initial topology with respect to the family of maps $\tau_U \colon \mathfrak{X}_{\mathrm{Orb}}\left(Q\right) \to \mathfrak{X}\left(U\right), [\hat{f}] \mapsto f_U, \ (U,G,\psi) \in \mathcal{U}$.

(b) Let $\mathcal{V} := \{\, (V_j, H_j, \psi_j) | j \in J \,\} \subseteq \mathcal{U}$ be a locally finite orbifold atlas, such that each chart in $\mathcal{V}$ is relatively compact. Endow $\mathfrak{X}_{\mathrm{Orb}}\left(Q\right)_c$ with the locally convex vector topology making

$$\Lambda_{\mathcal{V}} \colon \mathfrak{X}_{\mathrm{Orb}}\left(Q\right)_c \to \bigoplus_{j\in J} \mathfrak{X}\left(V_j\right), \ [\hat{f}] \mapsto (f_{V_j})_{j\in J}$$

from Lemma 4.3.2 (b) a topological embedding. We call this topology the *compactly supported orbisection topology* (or *c.s. orbisection topology*).
With respect to this topology, the linear maps $\tau_{V_j} \colon \mathfrak{X}_{\mathrm{Orb}}\left(Q\right)_c \to \mathfrak{X}\left(V_j\right), [\hat{f}] \mapsto f_{V_j}$ are continuous for each $(V_j, G_j, \psi_j) \in \mathcal{V}$.

[3] The article [25] use an other concept of differentiability in locally convex vector spaces which is adapted to non-discrete topological fields. However as [5, Proposition 7.4] asserts, this concept of differentiability coincides with the one from Definition 2.1.1 on open sets of locally convex vector spaces over the field $\mathbb{R}$. As we are only intersted in this case we may use the results of [25] without restriction.

As in the manifold case, it suffices to work with any atlas for the orbisection topology respectively the c.s. orbisection topology does not depend on the choice of locally finite atlas. To prove the independence of the compactly supported orbisection topology from the choice of the orbifold atlas relatively compact charts are needed. Hence the additional requirement in Definition 4.3.3 is needed.

4.3.4 Lemma *Let $\mathcal{W} = \{ (W_i, G_i, \phi_i) | i \in I \} \subseteq \mathcal{U}$ be an arbitrary orbifold atlas for Q.*

(a) *The orbisection topology is initial with respect to the family $(\tau_{W_i})_{(W_i, H_i, \phi_i) \in \mathcal{W}}$.*
(b) *Let $\mathcal{W}$ be locally finite such that each chart in $\mathcal{W}$ is relatively compact. The c.s. orbisection topology $\mathcal{O}_\mathcal{V}$ with respect to $\mathcal{V}$ and the c.s. orbisection topology $\mathcal{O}_\mathcal{W}$ with respect to $\mathcal{W}$ coincide.*

Proof. (a) Let $\mathcal{T}$ be the initial topology on $\mathfrak{X}_{\mathrm{Orb}}(Q)$ with respect to $(\tau_{W_i})_{(W_i, G_i, \phi_i) \in \mathcal{W}}$. The topology $\mathcal{T}$ is clearly coarser than the orbisection topology. Fix $(U, H, \psi) \in \mathcal{U}$, we have to show that $\tau_U \colon (\mathfrak{X}_{\mathrm{Orb}}(Q), \mathcal{T}) \to \mathfrak{X}(U)$ is a continuous map.
The family of open sets $\left\{ \tilde{V}_i := \tilde{U} \cap \tilde{W}_i \middle| i \in I \right\}$ is an open cover of $\tilde{U}$. Define $V_i := \psi^{-1}(\tilde{V}_i)$ to derive an open cover of U. By [25, Lemma F.16] the topology on $\mathfrak{X}(U)$ is initial with respect to the family $(\mathrm{res}^U_{V_i})_{i \in I}$. Since every V_i satisfies $\psi(V_i) \subseteq \phi_i(W_i)$ by compatibility of orbifold charts, there is a family of change of charts maps $(\lambda_{ik})_{k \in K_i} \subseteq \mathcal{C}h_{W_i, U}$, such that $\bigcup_{k \in K_i} \operatorname{cod} \lambda_{ik} = V_i$.
Another application of [25, Lemma F.16] implies that the topology of $\mathfrak{X}(V_i)$ is initial with respect to $(\mathrm{res}^{V_i}_{\operatorname{cod} \lambda_{ik}})_{k \in K_i}$. Using transitivity of initial topologies, τ_U will be continuous with respect to $\mathcal{T}$ if we can show that every

$$f_{ik} := \mathrm{res}^U_{\operatorname{cod} \lambda_{ik}} \circ \tau_U \colon \mathfrak{X}_{\mathrm{Orb}}(Q) \to \mathfrak{X}(\operatorname{cod} \lambda_{ik})$$

is continuous for $i \in I, k \in K_i$. But [25, Lemma F.15 (a)] implies that $\mathrm{res}^{W_i}_{\operatorname{dom} \lambda} \colon \mathfrak{X}(W_i) \to \mathfrak{X}(\operatorname{dom} \lambda_{ik})$ is continuous. Thus the proof of Lemma 4.3.2 shows that each $g_{\lambda_{ik}}$ is continuous. We conclude from $f_{ik} = g_\lambda \, \mathrm{res}^{W_i}_{\operatorname{dom} \lambda} \tau_{W_i}$ that τ_U is continuous with respect to $\mathcal{T}$ for every $(U, G, \psi) \in \mathcal{U}$. Thus $\mathcal{T}$ is finer than the orbisection topology, whence both topologies coincide.
(b) Consider $\mathcal{V} = \{ (V_j, H_j, \psi_j) | j \in J \}$. Combining Lemma 2.6.2 and Lemma 2.6.5 (c) we may choose a locally finite refinement $\mathcal{R}$ of $\mathcal{V}$ and $\mathcal{W}$ such that each chart in $\mathcal{R}$ is relatively compact, embedds into some charts in $\mathcal{V}$ and $\mathcal{W}$ and the closure of the embedded images are compact subsets of the orbifold charts. As we can interchange $\mathcal{V}$ and $\mathcal{W}$ it is clearly sufficient to prove that the c.s. orbisection topologies induced by $\mathcal{V}$ and $\mathcal{R}$ coincide. To keep the notation under control, without loss of generality $\mathcal{W}$ is a refinement of $\mathcal{V}$ with the properties described above (i.e. the ones from Lemma 2.6.5 (c)). In particular for each $i \in I$ the set W_i is a $H_{\alpha(i)}$-stable relatively compact subset of $V_{\alpha(i)}$.

Step 1: The topology $\mathcal{O}_\mathcal{W}$ is finer than $\mathcal{O}_\mathcal{V}$ Let $\alpha \colon I \to J$ be the map which assigns to $(W_i, G_i, \phi_i) \in \mathcal{W}$ the chart $(V_{\alpha(i)}, H_{\alpha(i)}, \psi_{\alpha(i)}) \in \mathcal{V}$, such that the inclusion $W_i \subseteq V_{\alpha(i)}$ is an embedding of orbifold charts. As the charts are relatively compact, by local finiteness of the atlases, $\alpha^{-1}(j)$ must be a finite set for each $j \in J$. Since $\alpha^{-1}(j)$ is finite, we obtain a

well-defined linear map

$$\rho_j\colon \mathfrak{X}(V_j) \to \bigoplus_{i\in I} \mathfrak{X}(W_i), X \mapsto \begin{cases} X_{|W_i} & \text{if } \alpha(i) = j \\ 0_{W_i} & \text{otherwise} \end{cases}.$$

For $\alpha^{-1}(j) = \emptyset$, ρ_j is continuous as a constant map. If $\alpha^{-1}(j) \neq \emptyset$, we may rewrite ρ_j as a composition:

$$\mathfrak{X}(V_j) \stackrel{\rho_j^*}{\to} \prod_{i\in\alpha^{-1}(j)} \mathfrak{X}(W_i) \hookrightarrow \bigoplus_{i\in I} \mathfrak{X}(W_i).$$

Here the second map is the canonical embedding and $\rho_j^* := (\mathrm{res}^{V_j}_{W_i})_{i\in\alpha^{-1}(j)}$ is the obvious combination of restriction maps. As each component of ρ_j^* into the direct product is continuous (see [25, Lemma F.15 (a)], the map is continuous and therefore ρ_j is continuous. We obtain a continuous linear map $\rho\colon \bigoplus_{j\in J} \mathfrak{X}(V_j) \to \bigoplus_{i\in I} \mathfrak{X}(W_i), (X_j)_{j\in J} \mapsto \sum_{j\in J} \rho_j(X_j)$. The topologies $\mathcal{O}_{\mathcal{V}}$ and $\mathcal{O}_{\mathcal{W}}$ are induced by the maps $\Lambda_{\mathcal{V}}\colon \mathfrak{X}_{\mathrm{Orb}}(Q)_c \to \bigoplus_{j\in J} \mathfrak{X}(V_j)$ resp. $\Lambda_{\mathcal{W}}\colon \mathfrak{X}_{\mathrm{Orb}}(Q)_c \to \bigoplus_{i\in I} \mathfrak{X}(W_i)$. Obviously ρ satisfies $\rho\circ\Lambda_{\mathcal{V}} = \Lambda_{\mathcal{W}}$ and thus $\mathcal{O}_{\mathcal{W}} \subseteq \mathcal{O}_{\mathcal{V}}$.

Step 2: Neighborhoods in $\mathfrak{X}(V_j)$ induce neighborhoods in $(\mathfrak{X}_{\mathrm{Orb}}(Q)_c, \mathcal{O}_{\mathcal{W}})$

The atlases $\mathcal{V}$ and $\mathcal{W}$ are locally finite and the charts in each one are relatively compact. Hence for each $i \in I$ resp. $j \in J$ there are finite sets $J_i \subseteq J$ (resp. $I_j \subseteq I$) such that

$$\phi_i(W_i) \cap \overline{\psi_k(V_k)} \neq \emptyset \text{ if and only if } k \in J_i,$$
$$\psi_j(V_j) \cap \overline{\phi_r(W_r)} \neq \emptyset \text{ if and only if } r \in I_j.$$

For $j \in J$ the topology on V_j is induced by an embedding $\Gamma_j\colon \mathfrak{X}(V_j) \to \prod_{k\in L_j} C^\infty(U_{\kappa_k}, \mathbb{R}^d)$, where (U_{κ_k}, κ_k) is a locally finite atlas of relatively compact manifold charts (this follows from Lemma C.3.2 by local compactness of the finite dimensional manifold V_j). Each open set in $\mathfrak{X}(V_j)$ may be obtained as $\Gamma_j^{-1}(U)$ for some open set $U \subseteq \prod_{k\in L_j} C^\infty(U_{\kappa_k}, \mathbb{R}^d)$. By definition of the product topology, a base of the topology is is given by the products of families of open sets, which differ from $C^\infty(U_{\kappa_k}, \mathbb{R}^d)$ only for finitely many k. Let U a set in the base of the product topology, such that $A_U \subseteq L_j$ is the finite set which indexes the components which differ from $C^\infty(U_{\kappa_k}, \mathbb{R}^d)$. To $U \subseteq \prod_{k\in L_j} C^\infty(U_{\kappa_k}, \mathbb{R}^d)$ associate the relatively compact set $P_U := \bigcup_{k\in A_U} U_{\kappa_k}$. We construct a neighborhood in $(\mathfrak{X}_{\mathrm{Orb}}(Q)_c, \mathcal{O}_{\mathcal{W}})$ induced by $V = \Gamma_j^{-1}(U) \subseteq \mathfrak{X}(V_j)$. By construction, there is an open set $\check{V} \subseteq \mathfrak{X}(P_U)$ such that $V = (\mathrm{res}^{V_j}_{P_U})^{-1}(\check{V})$ holds.

Consider the compact sets $K_U := \overline{P_U}$ and $\tilde{K}_U := \psi_j(K_U) \subseteq \psi_j(V_j)$. The set $\tilde{K}_U$ is covered by $\bigcup_{i\in I_j} \phi_i(W_i)$ since $\mathcal{W}$ is an atlas and $\phi_k(W_k) \cap \psi_j(V_j) \neq \emptyset$ iff $k \in I_j$. For each $i \in I_j$ we obtain compact sets $\tilde{K}_U \cap \psi_{\alpha(i)}(\overline{W_i}) \subseteq \psi_{\alpha(i)}(V_{\alpha(i)}) \cap \psi_j(V_j)$. Since the quotient map to an orbit space is a proper map, the set $K_U(i) := \psi_j^{-1}(\tilde{K}_U \cap \psi_i(\overline{W_i})) \cap K_U$ is compact. By construction these sets satisfy $K_U(i) \subseteq \psi_j^{-1}(\mathrm{Im}\,\psi_{\alpha(i)})$ and $K_U = \bigcup_{i\in I_j} K_U(i)$. Notice that $\phi_i(W_i) = \psi_{\alpha(i)}(W_i) \subseteq \psi_{\alpha(i)}(\overline{W_i})$ and $\psi_j(P_U) \subseteq \tilde{K}_U$ imply $P_U \cap \psi_j^{-1}(\mathrm{Im}\,\phi_i) \subseteq K_U(i)$.

Using compactness of $K_U(i)$, by compatibility of orbifold charts, there is a finite family $(\lambda^k_{ij})_{1\leq k\leq N_{U,i}}$ of embeddings of orbifold charts $\lambda^k_{ij}\colon V_j \supseteq \mathrm{dom}\,\lambda^k_{ij} \to V_{\alpha(i)}$ such that $K_U(i) \subseteq$

$\bigcup_{k=1}^{N_{U,i}} \operatorname{dom} \lambda_{ij}^k$. For any $x \in P_U \cap \psi_j^{-1}(\operatorname{Im} \phi_i) \subseteq K_U(i)$ there is some λ_{ij}^k with $x \in \operatorname{dom} \lambda_{ij}^k$. Since those maps are embeddings of orbifold charts, $\psi_{\alpha(i)} \lambda_{ij}^k(x) = \psi_j(x) \in \phi_i(W_i)$, hence $\lambda_{ij}^k(x) \in H_{\alpha(i)}.W_i$. Enlarge the family $(\lambda_{ij}^k)_k$:
Restrict each $\gamma \circ \lambda_{ij}^k$, $\gamma \in H_{\alpha(i)}$ to the open (possibly empty) subset of $O_{\gamma,k} \subseteq P_U \cap \psi_j^{-1}(\operatorname{Im} \phi_i)$ such that $\gamma \circ \lambda_{ij}^k(O_{\gamma,k}) \subseteq W_i$. Collecting all non-trivial maps, by finiteness of $H_{\alpha(i)}$ we obtain a finite family of open embeddings $\mu_{ij}^r \colon P_U \cap \psi_j^{-1}(\operatorname{Im} \phi_i) \supseteq \operatorname{dom} \mu_{ij}^r \to W_i,\ 1 \leq r \leq M_{U,i}$. As $P_U \cap \psi_j^{-1}(\operatorname{Im} \phi_i) \subseteq K_U(i)$, indeed $P_U \cap \psi_j^{-1}(\operatorname{Im} \phi_i) = \bigcup_{r=1}^{M_{U,i}} \operatorname{dom} \mu_{ij}^r$ holds.
The open sets $P_U \cap \psi_j^{-1}(\operatorname{Im} \phi_i)$, $i \in I_j$ cover P_U, whence the topology on $\mathfrak{X}(P_U)$ is initial w.r.t. the restriction maps $\operatorname{res}_{P_U \cap \psi_j^{-1}(\operatorname{Im} \phi_i)}^{P_U}$, $i \in I_j$ by [25, Lemma F.16]. For each $i \in I_j$ the topology on $\mathfrak{X}\left(P_U \cap \psi_j^{-1}(\operatorname{Im} \phi_i)\right)$ is initial with respect to the restriction maps $\operatorname{res}_{\operatorname{dom} \mu_{ij}^r}^{P_U \cap \psi_j^{-1}(\operatorname{Im} \phi_i)}$, $r = 1, \ldots, M_{U,i}$. By transitivity of initial topologies, the topology on $\mathfrak{X}(P_U)$ is initial with respect to the family of restrictions $\operatorname{res}_{\operatorname{dom} \mu_{ij}^r}^{P_U}, r = 1, \ldots, M_{U_i}, i \in I_j$. Oberve that each μ_{ij}^r arises as restriction of $\gamma.\lambda_{ij}^k$ to $O_{\gamma,k}$ for suitable $\gamma \in H_{\alpha(i)}$ and λ_{ij}^k. Hence the canonical lifts satisfy

$$\sigma_{W_i} \mu_{ij}^r = \sigma_{V_{\alpha(i)}}|_{W_i} (\gamma \lambda_{ij}^k)|_{O_{\gamma,k}} = T(\gamma \lambda_{ij}^k)|_{TO_{\gamma,k}} \sigma_{V_j}|_{O_{\gamma,k}} = T\mu_{ij}^r \sigma_{V_j}|_{\operatorname{dom} \mu_{ij}^r}. \tag{4.3.2}$$

Each μ_{ij}^r is an open embedding onto an open subset of W_i. Hence an argument as in the proof of Lemma 4.3.2 (a) shows that $\theta_{\mu_{ij}^r} \colon \mathfrak{X}\left(\operatorname{dom} \mu_{ij}^r\right) \to \mathfrak{X}\left(\operatorname{Im} \mu_{ij}^r\right), \sigma \mapsto T\mu_{ij}^r \circ \sigma \circ (\mu_{ij}^r)^{-1}$ is an isomorphism of topological vector spaces. Define $E_{ij} := \prod_{1 \leq r \leq M_{U_i}} \mathfrak{X}\left(\operatorname{cod} \mu_{ij}^r\right)$. Direct products in the category of topological vector spaces are functorial, whence we obtain an isomorphism $\prod_{i \in I_j} \prod_{1 \leq r \leq M_{U_i}} \mathfrak{X}\left(\operatorname{dom} \mu_{ij}^r\right) \cong \prod_{i \in I_j} E_{ij}$. The identity (4.3.2) implies, that the $\theta_{\mu_{ij}^r}$-image of the restriction of a canonical lift to $\operatorname{dom} \mu_{ij}^r$ is the restriction of a canonical lift of the same orbisection to $\operatorname{cod} \mu_{ij}^r$. The open sets $\operatorname{cod} \mu_{ij}^r$ are contained in W_i and by continuity of the restriction maps $\operatorname{res}_{\operatorname{cod} \mu_{ij}^r}^{W_i}$, open sets in E_{ij} induce open sets $\mathfrak{X}(W_i)$. The topology on $\mathfrak{X}(P_U)$ is initial with respect to the restrictions to the open sets $\operatorname{dom} \mu_{ij}^r$, $i \in I_j, 1 \leq r \leq M_{U_i}$. Summing up, for each $V = \Gamma_j^{-1}(U) = (\operatorname{res}_{P_U}^{V_j})^{-1}(\check{V})$ there is a family of open zero-neighborhoods $\hat{V}^i \subseteq \mathfrak{X}(W_i), i \in I_j$ with the following property: Let $[\hat{\sigma}] \in \mathfrak{X}_{\mathrm{Orb}}(Q)_c$ with canonical lift σ_{V_j} on V_j and σ_{W_i} on $W_i, i \in I_j$. Then σ_{V_i} is an element of V if and only if $\sigma_{W_i} \in \hat{V}^i\ \forall i \in I_j$.

Step 3: The countable case We shall assume for this step only that the atlases $\mathcal{V}, \mathcal{W}$ are indexed by countable sets I, J.
Consider the vector spaces $(\bigoplus_{i \in I} \mathfrak{X}(V_i))_{\mathrm{box}}$ respectively $(\bigoplus_{j \in J} \mathfrak{X}(V_j))_{\mathrm{box}}$ endowed with the box-topology. Since I, J are countable, the box topology coincides with the locally convex direct sum topology by [37, Proposition 4.1.4]. A typical neighborhood in $\bigoplus_{j \in J} \mathfrak{X}(V_j)$ is given by $U := \oplus_{j \in J} U_j$, where $U_j \subseteq \mathfrak{X}(V_j)$ is an open set. By Step 2 we may associate to each U_j a finite family $\hat{U}_j^i \subseteq \mathfrak{X}(W_i), i \in I_j$ and thus an open subset $\hat{U}_i := \bigcap_{j \in J_i} \hat{U}_j^i \subseteq \mathfrak{X}(W_i)$. The box $\hat{U} := \oplus_{i \in I} \hat{U}_i$ is an open set in $(\bigoplus_{i \in I} \mathfrak{X}(W_i))_{\mathrm{box}}$. Following the construction in step 2, we obtain the following conditions for an orbisection $[\hat{\sigma}]$ with families of canonical lifts $(\sigma_{W_i})_I$

with respect to $\mathcal{W}$ and $(\sigma_{V_j})_J$ with respect to $\mathcal{V}$:

$$[\hat{\sigma}] \in \Lambda_{\mathcal{V}}^{-1}(U) \Leftrightarrow \Lambda_{\mathcal{V}}([\hat{\sigma}]) \in U \Leftrightarrow (\forall j \in J) \;\; \sigma_{V_j} \in U_j \Leftrightarrow (\forall j \in J)(\forall i \in I_j) \;\; \sigma_{W_i} \in \hat{U}^i_j$$
$$\Leftrightarrow (\forall i \in I) \;\; \sigma_{W_i} \in \bigcap_{j \in J_i} \hat{U}^i_j \Leftrightarrow \Lambda_{\mathcal{W}}([\hat{\sigma}]) \in \hat{U} \Leftrightarrow [\hat{\sigma}] \in \Lambda_{\mathcal{W}}^{-1}(\hat{U})$$

This proves $\Lambda_{\mathcal{V}}^{-1}(U) = \Lambda_{\mathcal{W}}^{-1}(\hat{U})$, whence $\mathcal{O}_{\mathcal{V}} \subseteq \mathcal{O}_{\mathcal{W}}$ holds. In particular an isomorphism of topological vector spaces is given by $\rho|^{\operatorname{Im}\Lambda_{\mathcal{W}}}_{\operatorname{Im}\Lambda_{\mathcal{V}}} = \Lambda_{\mathcal{W}}^{|\operatorname{Im}\Lambda_{\mathcal{W}}} \circ (\Lambda_{\mathcal{V}}^{|\operatorname{Im}\Lambda_{\mathcal{V}}})^{-1})\colon \operatorname{Im}\Lambda_{\mathcal{V}} \to \operatorname{Im}\Lambda_{\mathcal{W}}$. If both atlases are countable this completes the proof, as a combination of Step 1 and Step 3 yields $\mathcal{O}_{\mathcal{V}} = \mathcal{O}_{\mathcal{W}}$.

Step 4: The general case In general neither $\mathcal{V}$ nor $\mathcal{W}$ will be countable (since the orbifolds we consider are **not** σ-compact). Orbifold charts are connected, whence each chart is contained in exactly one connected component. Let $\mathcal{C}$ be the family of connected components of Q and for $C \in \mathcal{C}$ and an atlas $\mathcal{A}$ define $\mathcal{A}_C := \{ (V,H,\psi) \in \mathcal{A} | \psi(V) \subseteq C \}$. The subset $\mathcal{A}_C$ is an atlas of orbifold charts for the component C. We may split the atlases $\mathcal{V}$, $\mathcal{W}$ into disjoint unions $\mathcal{V} = \bigsqcup_{C \in \mathcal{C}} \mathcal{V}_C$ resp. $\mathcal{W} = \bigsqcup_{C \in \mathcal{C}} \mathcal{W}_C$. By construction $\mathcal{W}_C$ still is a refinement of $\mathcal{V}_C$ with the properties described in Lemma 2.6.5 (c) for each $C \in \mathcal{C}$. Decompose the direct sums

$$\bigoplus_{i \in I} \mathfrak{X}(W_i) = \bigoplus_{C \in \mathcal{C}} \left(\bigoplus_{(W,G,\phi) \in \mathcal{W}_C} \mathfrak{X}(W) \right) \qquad \bigoplus_{j \in J} \mathfrak{X}(V_j) = \bigoplus_{C \in \mathcal{C}} \left(\bigoplus_{(V,H,\psi) \in \mathcal{W}_C} \mathfrak{X}(V) \right)$$

and observe that the maps $\Lambda_{\mathcal{V}}$ and $\Lambda_{\mathcal{W}}$ decompose as $\Lambda_{\mathcal{V}} = (\Lambda_{\mathcal{V}_C})_{C \in \mathcal{C}}$ and $\Lambda_{\mathcal{W}} = (\Lambda_{\mathcal{W}_C})_{C \in \mathcal{C}}$. Every connected component $C \subseteq Q$ is σ-compact by Proposition 2.4.3 (d). Since $\mathcal{W}_C$ and $\mathcal{V}_C$ are locally finite, both atlases have to be countable. Step 3 yields for each connected component $\mathcal{C}$ an isomorphism $\rho|^{\operatorname{Im}\Lambda_{\mathcal{W}_C}}_{\operatorname{Im}\Lambda_{\mathcal{V}_C}} = \Lambda_{\mathcal{W}_C}^{|\operatorname{Im}\Lambda_{\mathcal{W}_C}} (\Lambda_{\mathcal{V}_C}^{-1})^{|\operatorname{Im}\Lambda_{\mathcal{V}_C}} \colon \operatorname{Im}\Lambda_{\mathcal{V}_C} \to \operatorname{Im}\Lambda_{\mathcal{W}_C}$. Taking direct sums in the category of topological vector spaces is is functorial. Therefore $\oplus_{C \in \mathcal{C}} \rho_{|\operatorname{Im}\Lambda_{\mathcal{V}_C}} \colon \bigoplus_{C \in \mathcal{C}} \operatorname{Im}\Lambda_{\mathcal{V}_C} \to \bigoplus_{C \in \mathcal{C}} \operatorname{Im}\Lambda_{\mathcal{W}_C}$ is an isomorphism of locally convex topological vector spaces. Observe that the families of canonical inclusions (of vector subspaces) $\iota_{\mathcal{C}} \colon \operatorname{Im}\Lambda_{\mathcal{V}_C} \hookrightarrow \bigoplus_{(V,H,\psi) \in \mathcal{W}_C} \mathfrak{X}(V)$ respectively $\iota'_C \colon \operatorname{Im}\Lambda_{\mathcal{W}_C} \hookrightarrow \bigoplus_{(W,G,\phi) \in \mathcal{W}_C} \mathfrak{X}(W)$ induce continuous linear maps $\iota := \oplus_{C \in \mathcal{C}} \iota_C$ respectively. $\iota' := \oplus_{C \in \mathcal{C}} \iota'_C$. By [10, II.6, Prop. 8] the subspace topology on $\operatorname{Im}\iota$ turns ι into an isomorphism of topological vector spaces and the same holds for the subspace topology on $\operatorname{Im}\iota'$ and ι'. We deduce that

$$\rho|^{\operatorname{Im}\Lambda_{\mathcal{W}}}_{\operatorname{Im}\Lambda_{\mathcal{V}}} = \Lambda_{\mathcal{W}}|^{\operatorname{Im}\Lambda_{\mathcal{W}}} \circ (\Lambda_{\mathcal{V}}|^{\operatorname{Im}\Lambda_{\mathcal{V}}})^{-1} = \iota' \circ \bigoplus_{C \in \mathcal{C}} \left(\Lambda_{\mathcal{W}_C}|^{\operatorname{Im}\Lambda_{\mathcal{W}_C}} \circ (\Lambda_{\mathcal{V}_C}|^{\operatorname{Im}\Lambda_{\mathcal{V}_C}})^{-1} \right) \circ \iota^{-1}$$

is an isomorphism of topological vector spaces. Thus $\mathcal{O}_{\mathcal{V}} \subseteq \mathcal{O}_{\mathcal{W}}$ holds and by Step 1, we finally obtain $\mathcal{O}_{\mathcal{V}} = \mathcal{O}_{\mathcal{W}}$.

□

4.3.5 Theorem *Let $(Q,\mathcal{U})$ be a second countable orbifold, i.e. Q is a second countable space (or equivalently Q is a σ-compact space). The topological vector space $\mathfrak{X}_{Orb}(Q)$ is a Fréchet space.*

Proof. As Q is second countable, there is a countable orbifold atlas $\{(U_i, G_i, \psi_i)\}_{i\in\mathbb{N}}$ for Q. By Lemma 4.3.4 the orbisection topology is initial with respect to the maps

$$\tau_{U_i} : \mathfrak{X}_{\mathrm{Orb}}(Q) \to \mathfrak{X}(U_i), [\hat{f}] \mapsto f_{U_i}.$$

In particular Lemma 4.3.2 yields a linear topological embedding

$$\Lambda : \mathfrak{X}_{\mathrm{Orb}}(Q) \to \prod_{i\in\mathbb{N}} \mathfrak{X}(U_i), [\hat{f}] \mapsto (f_{U_i})_{i\in I}$$

onto a closed subspace. The manifolds U_i are finite dimensional, connected and paracompact manifolds. Thus by Proposition 2.4.2, every U_i is σ-compact and second countable. The space $\mathbb{R}^n$ is a Fréchet space over the locally compact field $\mathbb{R}$. Combining these observations with C.3.2 and [25, Proposition 4.19],$\mathfrak{X}(U_i)$ with the topology defined in definition C.3.1 is Fréchet space for each $i \in I$. The countable product of Fréchet spaces is a Fréchet space (cf. [19, IX. Corollary 7.3 and XIV. Theorem 2.5 (4)]) and thus $\prod_{i\in I} \mathfrak{X}(U_i)$ is a Fréchet space. Combining Lemma 4.3.2 and Lemma 4.3.4, $\mathfrak{X}_{\mathrm{Orb}}(Q)$ is homeomorphic to a closed subspace of the Fréchet space $\prod_{i\in I} \mathfrak{X}(U_i)$. Thus $\mathfrak{X}_{\mathrm{Orb}}(Q)$ is a Fréchet space. □

4.3.6 Corollary

(a) *The spaces $\mathfrak{X}_{Orb}(Q)$ with the orbisection topology and $\mathfrak{X}_{Orb}(Q)_c$ with the c.s. orbisection topology are Hausdorff topological vector spaces.*
(b) *If $(Q,\mathcal{U})$ is a compact orbifold, then the locally convex vector spaces $\mathfrak{X}_{Orb}(Q)$ and $\mathfrak{X}_{Orb}(Q)_c$ coincide. If Q is compact both spaces are Fréchet spaces.*
(c) *Let $\mathcal{V}$ be a locally finite orbifold atlas for Q. The family $\{\tau_V | (V, G, \psi) \in \mathcal{V}\}$ as in Definition 4.3.3 (b) forms a patchwork for $\mathfrak{X}_{Orb}(Q)_c$, turning it into a patched locally convex space. The topological embedding is given by $\Lambda_{\mathcal{C}}$ (cf. Definition C.3.4).*

Proof. (a) We endow the space of vector fields on a finite dimensional manifold with the topology introduced in Definition C.3.1. Since direct products and direct sums of Hausdorff locally convex vector spaces are again such spaces by [37, Proposition 4.3.3], the assertion follows from [25, Remark F.8] by definition of the topology.
(b) For finite index sets products and direct sums are canonically isomorphic. As locally finite coverings of compact spaces are finite together with 4.3.5 this proves the claim.
(c) Follows directly from the definition of the compactly supported orbisection topology 4.3.3.

□

4.3.7 Lemma *Let $K \subseteq Q$ be a compact subset and endow $\mathfrak{X}_{Orb}(Q)_K \subseteq \mathfrak{X}_{Orb}(Q)_c$ with the subspace topology. The space $\mathfrak{X}_{Orb}(Q)_K$ is a closed subspace of $\mathfrak{X}_{Orb}(Q)_c$.*

Proof. Choose an arbitrary locally finite orbifold atlas $\mathcal{V} := \{(V_i, G_i, \psi_i) \in \mathcal{U} | i \in I\}$ for $(Q,\mathcal{U})$. By Lemma 4.3.4 (b), there is a topological embedding $\Lambda_{\mathcal{V}} : \mathfrak{X}_{\mathrm{Orb}}(Q)_c \to \bigoplus_{i\in I} \mathfrak{X}(V_i)$, whose image is

closed. For each $i \in I$ we obtain a (possibly empty) subset $U_i := \psi_i^{-1}(Q \setminus K)$. If $U_i = \emptyset$ holds, define $A_i := \mathfrak{X}(V_i)$. Otherwise consider $x \in U_i$ and a manifold chart (W_ψ, ψ) for V_i such that $x \in W_\psi$. The evaluation map $\mathrm{ev}_x^\psi \colon C^\infty(W_\psi, \mathbb{R}^d) \to \mathbb{R}^d, \xi \mapsto \xi(x)$ is continuous by [25, Proposition 11.1]. As the topology on $\mathfrak{X}(V_i)$ is initial with respect to the maps $\theta_\psi \colon \mathfrak{X}(V_i) \to C^\infty(W_\psi, \mathbb{R}^d), X \mapsto X_\psi$ the point evaluation $\mathrm{ev}_x \colon \mathfrak{X}(V_i) \to \mathbb{R}^d, \sigma \mapsto \mathrm{ev}_x^\psi \circ \theta_\psi(\sigma)$ is continuous. Hence we derive a closed set $A_i := \bigcap_{x \in U_i} \mathrm{ev}_x^{-1}(0)$. From [10, II.5 Cor. 1] we conclude that $A := \oplus_{i \in I} A_i = \prod_{i \in I} A_i \cap \bigoplus_{i \in I} \mathfrak{X}(V_i)$ is closed. By construction each orbisection in $\Lambda_{\mathcal{V}}^{-1}(A)$ vanishes off K, whereas its support must be contained in K. We deduce $\Lambda_{\mathcal{V}}^{-1}(A) = \mathfrak{X}_{\mathrm{Orb}}(Q1)_K$, whence $\mathfrak{X}_{\mathrm{Orb}}(Q)_K$ is a closed set. □

The results in this section suggest that orbisections behave in many ways as vector fields for finite dimensional manifolds. Before we end this section we have to point out that in some ways orbisections do *not* behave like vector fields. There may be formal orbifold tangent vectors which are *not contained* in the image of any orbisection. In the manifold case this may never occur. The following example was first considered by Borzellino et al. (see [6, Example 43]) in the context of their notion of orbifold maps:

4.3.8 Example Consider $\mathbb{R}$, with an action induced by the linear diffeomorphism $\gamma \colon \mathbb{R} \to \mathbb{R}, x \mapsto -x$. Set $G := \langle \gamma \rangle$ and let $\psi \colon \mathbb{R} \to \mathbb{R}/G$ be the quotient map to the orbit space. The quotient is isomorphic to $Q := [0, \infty[$ (as a subspace of $\mathbb{R}$). By abuse of notation we obtain an orbifold atlas $\mathcal{U} := \{ (\mathbb{R}, G, \psi) \}$ for Q. Now $(Q, \mathcal{U})$ is an orbifold and the local groups are trivial for every point except 0 (where it is isomorphic to G). We may thus compute the tangent spaces of Q at $x \in Q$ in the following way:
For $x \neq 0$ we have $\mathcal{T}_x Q \cong \mathbb{R}$ and $\mathcal{T}_0 Q \cong [0, \infty[$. An atlas for the tangent orbibundle is induced by the orbifold chart $(T\mathbb{R}, G, T\psi)$, where G acts on $T\mathbb{R}$ via the derived action. Taking identifications we obtain $T\mathbb{R} \cong \mathbb{R}^2$. The group G acts via elements of $O(1)$ on $\mathbb{R}$. Hence its action on $T\mathbb{R}$ is induced by the linear map $T\gamma \colon \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (-x, -y)$. The topological base space of the tangent orbibundle is thus $\mathcal{T}Q = \mathbb{R}^2/G$. The zero vector is the only fixed point of the derived action of G. Since orbisections preserve local groups by Proposition 4.2.5, every orbisection maps $0 \in Q$ to $0 \in \mathbb{R}^2/G \cong \mathcal{T}Q$. Thus all orbisections in $\mathfrak{X}_{\mathrm{Orb}}(Q)$ must vanish in $0 \in Q$ and

$$Q' := \bigcup_{(f, \{ \hat{f}_{(\mathbb{R}, \mathbb{Z}_2, \psi)} \}, P, \nu) \in \mathfrak{X}_{\mathrm{Orb}}(Q)} \operatorname{Im} f \subsetneq \mathcal{T}Q$$

Is the topological subspace Q' at least an orbifold? We shall prove that the answer to this question is negative. Indeed it will turn out that Q' is not locally compact.

Claim: $Q' \cong \{ (0,0) \} \cup \{]0, \infty[\times \mathbb{R} \} \subsetneq \mathbb{R}^2$ with the subspace topology. If this were true, then the assertion follows from the next argument: In the subspace topology on Q' a neigborhoods base of $(0,0)$ is given by $W_\varepsilon = B_\varepsilon^{\mathbb{R}^2}(0) \cap Q'$, where $\varepsilon > 0$ is arbitrary. Arguing indirectly, we assume that $(0,0)$ has some compact neighborhood $C_{(0,0)}$ in Q'. There is an $\varepsilon > 0$ such that $W_\varepsilon \subseteq C_{(0,0)}$. Hence $\overline{W_\varepsilon}$ has to be a compact set. Choose a sequence converging to some $(0, r) \in B_\varepsilon^{\mathbb{R}^2}(0) \subseteq \mathbb{R}^2$, $r \neq 0$, then this sequence contains no subsequence which converges in Q'. This contradicts the compactness

of $\overline{W_\varepsilon}$, since Q' is a metric space. Therefore Q' is not locally compact, whence it may not support an orbifold structure by Proposition 2.4.3.
Proof of the claim: Remark 4.2.10 shows that every orbisection is uniquely determined by a vector field $\xi \in \mathfrak{X}(\mathbb{R})$, such that $T\gamma\xi = \xi \circ \gamma$ holds. In other words: $\forall x \in \mathbb{R},\ -\xi(x) = \xi(-x)$. A vector field in $\mathfrak{X}(\mathbb{R})$ is uniquely determined as $\xi = (\mathrm{id}_\mathbb{R}, f_\xi)$ for some smooth map $f_\xi \colon \mathbb{R} \to \mathbb{R}$. Thus an orbisection is uniquely determined by its associated smooth map. We will show first that every tangent space $\mathcal{T}_pQ$ for $p \neq 0$ is contained in Q'. Obviously this will be true, if for every pair $(x, v) \in \mathbb{R} \setminus \{0\} \times \mathbb{R}$, there is a smooth map $F_v \colon \mathbb{R} \to \mathbb{R}$, such that

$$F_v(x) = v \text{ and } F_v(-w) = -F_v(w) \quad \forall w \in \mathbb{R} \tag{4.3.3}$$

holds. We now construct such a map for arbitrary $(x, v) \in \mathbb{R} \setminus \{0\} \times \mathbb{R}$:
Choose an arbitrary smooth bump function $h \colon \mathbb{R} \to [0, 1]$, such that $h(x) = 1$ and $\operatorname{supp} h \subseteq]0, \infty[$ if $x > 0$ resp. $\operatorname{supp} h \subseteq]-\infty, 0[$ if $x < 0$ (a smooth function with these properties exists by [36, Sect. 2.2]). Define the smooth function

$$F_v \colon \mathbb{R} \to \mathbb{R}, t \mapsto (h(t) - h(-t)) \cdot v.$$

Then $F_v(x) = v = -F_v(-x)$ and $F_v(-t) = -F_v(t)$ holds for each $t \in \mathbb{R}$. Hence for $p \in Q \setminus \{0\}$ the inclusion $\mathcal{T}_pQ \subseteq Q'$ holds. A computation proves $([0, \infty[\times\mathbb{R})/\sim\, = T\psi([0, \infty[\times\mathbb{R}) = \mathbb{R}^2/G$, where $x \sim x'$ if and only if both elements are in the same G-orbit: $x = T\gamma.x'$ (respectively $x = x'$).
The map $T\psi$ restricts to a continuous surjective map $q := T\psi|_{[0,\infty[\times\mathbb{R}}$ on $[0, \infty[\times\mathbb{R} \subseteq \mathbb{R}^2$. By Lemma B.1.4 (b) $T\psi$ is a closed map, whence we deduce from [19, III. Theorem 11.4] for $A \subseteq [0, \infty[\times\mathbb{R}$:

$$\overline{q(A)} = \overline{T\psi(A)} \subseteq T\psi(\overline{A}) = q(\overline{A})$$

The last identity follows since the closure $\overline{A}$ with respect to $\mathbb{R}^2$ is contained in the closed subset $[0, \infty[\times\mathbb{R}$. Another application of [19, III. Theorem 11.4] shows that q is a closed mapping. Furthermore $q^{-1}(Q') = \{(0, 0)\} \cup]0, \infty[\times\mathbb{R}$ holds. The closed quotient map q restricts to a quotient map $q|^{Q'}_{q^{-1}(Q')}$ by [19, VI. Thm. 2.1]. This restriction of q is an injective quotient map. Hence we obtain a homeomorphism $Q' \cong \{(0, 0)\} \cup]0, \infty[\times\mathbb{R} \subsetneq \mathbb{R}^2$, thus proving the claim.

5. Riemannian Geometry on Orbifolds

In this section the notion of a Riemannian orbifold metric is introduced. Our approach follows the construction of Riemannian metrics on manifolds (cf. [17, Ch. 1.2, Proposition 2.10]). The corresponding construction of such an object for an orbifold is well known (see for example [48, Proposition 2.20], we also recommend the survey in [14, Appendix 4.2]). Nevertheless the results are repeat here for the readers convenience and to fix some notation.

5.0.1 Definition (Riemannian orbifold metric) Let $(Q,\mathcal{U})$ be an orbifold and consider some orbifold atlas $\mathcal{V} := \{ (V_i, G_i, \psi_i) | i \in I \}$ for $(Q,\mathcal{U})$. A *Riemannian orbifold metric* on Q is a collection $\rho = (\rho_i)_{i \in I}$, where ρ_i is a Riemannian metric on the manifold V_i, such that the following holds:

(Compatibility) For $(i,j) \in I \times I$, and each open G_i-stable subset $S \subseteq V_i$, every embedding of orbifold charts $\lambda\colon (S, (G_i)_S, \psi_i|_S) \to (U_j, G_j, \psi_j)$ is a Riemannian embedding, i.e.

$$\rho_j(T_x\lambda(v), T_x\lambda(w)) = \rho_i(v,w), \quad \forall v, w \in T_xV_i,\ x \in S.$$

Let $(Q,\mathcal{U})$ be an orbifold endowed with a Riemannian orbifold metric ρ. The triple $(Q,\mathcal{U},\rho)$ is called *Riemannian orbifold*.

5.0.2 Remark Consider a Riemannian orbifold metric ρ on some orbifold $(Q,\mathcal{U})$. For a chart $(V, G, \psi) \in \mathcal{V}$ the group G acts by open self-embeddings of orbifold charts. If V is endowed with a member ρ_V of ρ, each element of G thus acts as a Riemannian isometry with respect to ρ_V.

5.0.3 Proposition ([48, Proposition 2.20]) *An orbifold $(Q,\mathcal{U})$ admits a Riemannian orbifold metric ρ.*

Proof. By Lemma 2.6.5, there is a locally finite representative $\mathcal{V} := \{ (V_i, G_i, \psi_i) | i \in I \}$ of $\mathcal{U}$. Let $\{ \hat{\chi}_i \}_{i \in I}$ be a smooth orbifold partition of unity subordinate to $\mathcal{V}$, which exists due to Proposition 3.3.2. Recall from 3.3.3 that for every pair $(i,j) \in I \times I$, there is a smooth lift $\chi_{i,j}$ of χ_i to (V_j, G_j, ψ_j). For $i \in I$ choose some Riemannian metrics $m^{(i)}$ on V_i (cf. [43, VII., §1, Proposition 1.1]). As G_i acts by diffeomorphisms, we obtain a pullback metrics on V_i. Averaging over G_i, on every tangent space there is a positive definite bilinear form:

$$\langle v, w \rangle_p^{(i)} := \frac{1}{|G_i|} \sum_{g \in G_i} m_{g.p}^{(i)}(T_pg.v, T_pg.w), \quad \forall v, w \in T_pV_i,\ p \in V_i$$

such that the family $\langle -, - \rangle^{(i)} := (\langle -, - \rangle_p^{(i)})_{p \in V_i}$ defines a Riemannian metric on V_i. By construction each element of G_i is a Riemannian isometry with respect to $\langle -, - \rangle^{(i)}$.
Define a Riemannian metric ρ_i on V_i as follows: The atlas $\mathcal{V}$ is locally finite, whence $\phi_i(p)$ with $p \in V_i$ is contained only in finitely many members of $\mathcal{V}$. Therefore there is an open G_i-stable subset $p \in S_p \subseteq V_i$ such that for $y \in S_p$, $\psi_i(y) \in \operatorname{supp} \chi_k$ holds only if $\psi_i(p) \in \operatorname{supp} \chi_k$ for $k \in I$. Shrinking

S_p, without loss of generality for each $k \in I$ with $\psi_i(p) \in \operatorname{supp} \chi_k$ there is an embedding of orbifold charts $\lambda_k^p \colon (S_p, (G_i)_{S_p}, \psi_i|_{S_p}) \to V_k$. If $\psi_i(p) \notin \operatorname{supp} \chi_k$ simply set $\lambda_k^p \equiv 0$ and define for $v, w \in T_pV_i$:

$$(\rho_i)_p(v,w) := \sum_{j \in I} \chi_{j,i}(p) \cdot \langle T_p\lambda_j^p(v), T_p\lambda_j^p(w)\rangle^{(j)}_{\lambda_j(p)}$$

Since the $\chi_{j,i}$ are the lifts of an orbifold partition of unity, $(\rho_i)_p$ is a well defined positive definite bilinear map on $T_pV_i \times T_pV_i$. The definition of $(\rho_i)_p$ neither depends on S_p nor on the choice of λ_k^p: To prove this claim, consider another G_i stable set $p \in S'_p$ with embeddings μ_k^p. Since we are only interested in the tangential map at p (which may be computed in an arbitrarily small open subset), we restrict μ_k^p and λ_k^p to an open and G_i-stable subset $S \subseteq S_p \cap S'_p$ which contains p. Proposition 2.2.2 (d) implies that there is a group element $g \in G_k$, such that $\mu^p_{k|S} = g.\lambda^p_{k|S}$. By construction every $g \in G_k$ is a Riemannian isometry with respect to $\langle -, - \rangle^{(k)}$. Thus every choice induces the same map.
The maps λ_j^p, $\chi_{k,i}$ are smooth and for each $k \in I$, $\langle -, - \rangle^{(k)}$ is a Riemannian metric, thus the family $\rho_i := ((\rho_i)_p)_{p \in V_i}$ defines a smooth map on each open set $TS_p \oplus TS_p \subseteq TV_i \oplus TV_i$. By construction the map does not depend on the set S_p and thus ρ_i is smooth on $TV_i \oplus TV_i$. Hence it is a Riemannian metric on V_i.
We claim that the family $(\rho_i)_{i \in I}$ satisfies the compatibility condition of Definition 5.0.1: Consider arbitrary $i, j \in I$ together with an open G_i-stable subset $S \subseteq V_i$ and an embedding of orbifold charts $\mu \colon (S, (G_i)_S, \psi_{i|S}) \to (V_j, G_j, \psi_j)$. For $p \in S$ and $v, w \in T_pV_i$ we have to show that $(\rho_j)_{\mu(p)}(T_p\mu(v), T_p\mu(w))$ coincides with $(\rho_i)_p(v,w)$.
Since μ is an embedding of orbifold charts and by construction one has $\chi_{k,j} = \chi_k \circ \psi_j$, we derive $\chi_{k,j} \circ \lambda = \chi_{k,i}|_{\operatorname{dom} \lambda}$. We compute:

$$\begin{aligned}
(\rho_j)_{\mu(p)}(T_p\mu(v), T_p\mu(w)) &= \sum_{k \in I} \chi_{k,j}(\mu(p)) \cdot \langle T_{\mu(p)}\lambda_k^{\mu(p)} T_p\mu(v), T_{\mu(p)}\lambda_k^{\mu(p)} T_p\mu(w)\rangle^{(k)}_{\lambda_k^{\mu(p)}\mu(p)} \\
&= \sum_{k \in I} \chi_{k,i}(p) \cdot \langle T_p(\underbrace{\lambda_k^{\mu(p)}\mu}_{\theta_k^p :=})(v), T_p(\underbrace{\lambda_k^{\mu(p)}\mu}_{=\theta_k^p})(w)\rangle^{(k)}_{\lambda_k^{\mu(p)}\mu(p)} \\
&= \sum_{k \in I} \chi_{k,i}(p) \cdot \langle T_p\theta_k^p(v), T_p\theta_k^p(w)\rangle^{(k)}_{\theta_k^p(p)}
\end{aligned}$$

Restrict every non zero map θ_k^p to a small open G_i-stable neighborhood of p, such that the restriction of θ_k^p yields an embedding of orbifold charts (cf. [48, Proposition 2.13]). As the defintion of the metric does not depend on the choice of embedding, indeed we obtain

$$(\rho_j)_{\mu(p)}(T_p\mu(v), T_p\mu(w)) = (\rho_i)_p(v,w).$$

The family ρ is compatibel as in Definition 5.0.1, whence it is a Riemannian orbifold metric. □

A Riemannian orbifold metric (uniquely) extends to the maximal orbifold atlas:

5.0.4 Proposition *Let $(Q, \mathcal{U})$ be an orbifold and $\mathcal{V} := \{ (V_i, G_i, \psi_i) | i \in I \}$ some representative of $\mathcal{U}$, for which there is a Riemannian orbifold metric $\rho = (\rho_i)_{i \in I}$. There exists a unique Riemannian orbifold metric $\hat{\rho}$, which extends ρ to $\mathcal{U}$.*

Proof. If $\mathcal{V}$ is not locally finite, replace $\mathcal{V}$ by a locally finite atlas; With Lemma 2.6.5 (b) choose a locally finite refinement $\mathcal{W}$ of $\mathcal{V}$. Then for every chart in $\mathcal{W}$, there is an embedding into some chart in $\mathcal{V}$ and we may endow the chart with the pullback metric with respect to this embedding. Since ρ is a Riemannian orbifold metric, the compatibility condition assures that:

i) the pullback metric does not depend on the choice of embedding
ii) the family of pullback metrics $\check{\rho}$ is compatible in the sense of defintion 5.0.1 and thus $\check{\rho}$ is a Riemannian orbifold metric.

Insteas of $\mathcal{V}$ the following construction uses the locally finite atlas $\mathcal{W} := \{ (W_j, G_j, \psi_j) | j \in J \}$ together with the induced Riemannian orbifold metric $\check{\rho}$. The construction will just return the Riemannian metrics on members of $\mathcal{V}$:
As $\mathcal{W}$ is locally finite, let $\{ \chi_j \}_{j \in J}$ be the orbifold partition of unity subordinate to $\mathcal{W}$. Consider $(U, H, \phi) \in \mathcal{U}$. We construct a Riemannian metric ρ_U on U, which is compatible with the family $\check{\rho}$ in the sense of Definition 5.0.1. For $p \in U$ there are only finitely many $j \in I$ such that $\phi(Q) \in \operatorname{supp} \chi_j$. Hence there is an an open H-stable subset $q \in S_q \subseteq U$, together with embedding of orbifold charts $\tau_j^q \colon (S_q, H_{S_q}, \phi_{|S_q}) \to (W_j, G_j, \psi_j)$, where $j \in J$ is an index such that $\phi(q) \in \operatorname{supp} \chi_j$ holds. If $\phi(q) \notin \operatorname{supp} \chi_j$, formally define $T_q \tau_j^q \equiv 0$. For $q \in U$ we define a positive definite bilinear map:

$$(\hat{\rho}_U)_q(v, w) = \sum_{j \in J} \chi_{j,j} \circ \tau_j^q(q) \cdot \check{\rho}_k(T_q \tau_j^q(v), T_q \tau_j^q(w)), \quad v, w \in T_q U$$

An argument analogous to the proof of Proposition 5.0.3 shows that $\hat{\rho}_U := ((\hat{\rho}_U)_p)_{p \in U}$ defines a Riemannian metric on U. We obtain a family of Riemannian metrics for $\mathcal{U}$ via $\hat{\rho} := \{ \rho_U | (U, H, \phi) \in \mathcal{U} \}$. Again as in the proof of Proposition 5.0.3 it is easy to see that the family $\hat{\rho}$ satisfies the compatibility condition of definition 5.0.1 and is thus an Riemannian orbifold metric.
We have to verify, that on every chart $(V, G, \psi) \in \mathcal{V}$, the metrics $\hat{\rho}_V$ and ρ_V coincide. By construction every $\check{\rho}_k$ is a pullback metric with respect to an embedding $\mu_k \colon (W_k, G_k, \psi_k) \to (V_{\sigma(k)}, G_{\sigma(k)}, \psi_{\sigma(k)}) \in \mathcal{V}$ for a suitable $\sigma(k) \in I$. A computation shows for $q \in V$, $v, w \in T_q V$:

$$\begin{aligned}
(\hat{\rho}_V)_q(v, w) &= \sum_{j \in J} \chi_{j,j} \circ \tau_j^q(q) \cdot (\check{\rho}_k)_{\tau_j^q(q)}(T_q \tau_j^q(v), T_q \tau_j^q(w)) \\
&= \sum_{j \in J} \chi_{j,j} \circ \tau_j^q(q) \cdot (\rho_{V_{\sigma(k)}})_{\mu_k \tau_j^q(k)}(T_q(\mu_k \tau_j^q)(v), T_q(\mu_k \tau_j^q)(w)) \\
&= \sum_{j \in J} \chi_j(\psi(q)) \cdot (\rho_V)_q(v, w) = (\rho_V)_q(v, w) \underbrace{\sum_{j \in J} \chi_j(\psi(q))}_{=1}
\end{aligned}$$

The third identity needs an explanation: Note that all non-zero maps $\mu_k \tau_j^q$ are open embeddings, respectively zero if and only if $\chi_j \phi(q) = 0$ holds. Since $(\rho_{V_j})_{j \in I}$ is a Riemannian orbifold metric, every open embedding of orbifold charts in $\mathcal{V}$ is a Riemannian isometry. Thus if $\mu_k \tau_j^q$ is an open embedding, we have $(\rho_{V_{\sigma(k)}})_{\mu_k \tau_j^q(k)}(T_q(\mu_k \tau_j^q)(v), T_q(\mu_k \tau_j^q)(w)) = (\rho_V)_q(v, w)$. Every other entry vanishes in the above sum. Since this only happens if $\chi_j \psi(q) = 0$, the desired formula holds. In conclusion we have shown $\rho_V = \hat{\rho}_V$.
The compatibility condition of a Riemannian orbifold metric now implies that $\hat{\rho}$ is unique. $\square$

Instead of defining a Riemannian orbifold metric as in Definition 5.0.1, Proposition 5.0.4 yields an equivalent definition of a Riemannian orbifold metric: It may be defined as a family of Riemannian metrics on the maximal atlas, which satisfies the compatibility condition (cf. [48, p.41]).
Either way, a Riemannian orbifold metric was defined using open embeddings of orbifold charts. The reader may have noticed that our working definition of orbifolds (cf. Definition 2.3.1) uses change of charts (but is of course equivalent to using embeddings of orbifold charts). The definitions in this chapter are slightly easier to formulate using open embeddings of orbifold charts, whereas we chose this approach. Nevertheless, change of orbifold charts are Riemannian isometries:

5.0.5 Lemma *Let $(Q,\mathcal{U},\rho)$ be a Riemannian orbifold and consider a change of orbifold charts $\lambda\colon U \supseteq \operatorname{dom}\lambda \to \operatorname{cod}\lambda \subseteq V$ for some $(U,H,\phi),(V,G,\psi) \in \mathcal{U}$. Furthermore let $\rho_{\operatorname{dom}\lambda}$ be the pullback metric of ρ_U with respect to the inclusion $\operatorname{dom}\lambda \subseteq U$. Then $\lambda\colon (\operatorname{dom}\lambda, \rho_{\operatorname{dom}\lambda}) \to (V,\rho_V)$ is a Riemannian embedding.*

Proof. Let $p \in \operatorname{dom}\lambda$ be arbitrary and choose an open connected H-stable subset $p \in S \subseteq \operatorname{dom}\lambda$. Then $(S, H_S, \phi_{|S})$ is an orbifold chart and $\lambda_{|S}$ is an embedding of orbifold charts. Since ρ_U and ρ_V are members of ρ, $\lambda|_S$ is a Riemannian embedding. In particular $(\rho_{\operatorname{dom}\lambda})_p = (\lambda^*\rho_V)_{\lambda(p)}$ holds. Since $p \in \operatorname{dom}\lambda$ was arbitrary λ is a Riemannian embedding. $\square$

5.0.6 Definition Let $(Q_i,\mathcal{U}_i,\rho_i), i = 1,2$ be Riemannian orbifolds and consider a map of orbifolds $[\hat{f}] \in \mathbf{Orb}((Q_1,\mathcal{U}_1),(Q_2,\mathcal{U}_2))$. The map $[\hat{f}]$ is called *orbifold isometric*, if there is a representative $\hat{f} = (f, \{ f_i | i \in I \}, P, \nu) \in \operatorname{Orb}(\mathcal{V},\mathcal{W})$, such that each lift $f_i\colon V_i \to W_{\alpha(i)}$ is an isometric immersion of the Riemannian manifold $(V_i, \rho_{1,i})$ into the Riemannian manifold $(W_{\alpha(i)}, \rho_{2,i})$.
If $[\hat{f}]$ is a diffeomorphism of orbifolds which is orbifold isometric, $[\hat{f}]$ is called *orbifold isometry*.

5.0.7 Remark The condition to be an isometric immersion of Riemannian manifolds may be checked locally. Lemma 5.0.5 (i.e. the compatibility conditions of Riemannian orbifold metrics) combined with Proposition 5.0.4 that a map $[\hat{f}]$ will be orbifold isometric if and only if each representative $\hat{f} := (f, \{ f_j \}_J, P, \nu)$ shares this property, that the family of lifts $\{ f_j \}_J$ consists of isometric immersions.
In view of Corollary 3.1.12 (d), a map $[\hat{f}]$ will be an orbifold isometry, if and only if there is a representative $\hat{f} := (f, \{ f_j \}_J, P, \nu)$ of $[\hat{f}]$, such that each member of $\{ f_j \}_J$ is a Riemannian isometry.
As an obvious first example, we mention that for a Riemannian orbifold $(Q,\mathcal{U},\rho)$ the identity morphism $\mathrm{id}_{(Q,\mathcal{U})}$ is an orbifold isometry.

5.0.8 Lemma *Let $(Q,\mathcal{U},\rho)$ be a Riemannian orbifold and $(Q_1,\mathcal{U}_1)$ be an orbifold together with an orbifold diffeomorphism $[\hat{f}] \in \mathbf{Orb}((Q_1,\mathcal{U}_1),(Q,\mathcal{U}))$. There exists a unique Riemannian orbifold metric $[\hat{f}]^*\rho$ on $(Q_1,\mathcal{U}_1)$ such that $[\hat{f}]$ becomes an orbifold isometry with respect to $(Q_1,\mathcal{U}_1,[\hat{f}]^*\rho)$ and $(Q,\mathcal{U},\rho)$. The Riemannian orbifold metric $[\hat{f}]^*\rho$ is called* pullback metric *induced by $[\hat{f}]$.*

Proof. Following Corollary 3.1.12 (d), we choose orbifold atlases $\mathcal{V} = \{ (V_i, G_i, \psi_i) \in \mathcal{U}_1 | i \in I \}$ and

$\mathcal{W} = \{ (W_j, H_j, \varphi_j) \in \mathcal{U} | j \in J \}$, such that there is a representative $\hat{g} = (f, \{ f_i | i \in I \}, P, \nu)$ of $[\hat{f}]$ with the following properties:

(a) Each $f_i \colon V_i \to W_{\beta(i)}$ in $\{ f_i | i \in I \}$ is a diffeomorphism,
(b) the map $\beta \colon I \to J$ is bijective,
(c) $P = \mathcal{C}h_{\mathcal{V}}$ holds and for $\lambda \in \mathcal{C}h_{V_i,V_j}$, one has $\nu(\lambda) = f_j \lambda f_i^{-1}|_{f_i(\operatorname{dom}\lambda)}$ by Corollary 3.1.8.

Proposition 5.0.4 yields a unique family of compatible Riemannian metrics $\{ \rho_j | j \in J \}$ induced by ρ, such that each chart (W_j, H_j, φ_j) turns into a Riemannian manifold (W_j, ρ_j). Endow each manifold V_i with the pullback metric $f_i^* \rho_{\beta(i)}$, turning f_i into a Riemannian isometry.
Claim: The family $\left\{ f_i^* \rho_{\beta(i)} \middle| i \in I \right\}$ turns each $\lambda \in \mathcal{C}h_{V_i,V_j}$, $i, j \in I$ into a Riemannian embedding. An argument analogous to the proof of Lemma 3.1.9 (c) shows that $\mu := f_j \lambda f_i^{-1}|_{f_i(\operatorname{dom}\lambda)} \in \mathcal{C}h_{W_{\beta(i)},W_{\beta(j)}}$ and $f_j \lambda = \mu f_i|_{\operatorname{dom}\lambda}$ holds. Consider $p \in \operatorname{dom}\lambda$ and compute for $v, w \in T_p V_i$:

$$\begin{aligned}
(f_j^* \rho_{\beta(j)})_{\lambda(p)}(T_p\lambda(v), T_p\lambda(w)) &= (\rho_{\beta(j)})_{f_j\lambda(p)}(T_p f_j \lambda(v), T_p f_j \lambda(w)) \\
&= (\rho_{\beta(j)})_{\mu f_i(p)}(T_p \mu f_i(v), T_p \mu f_i(w)) \\
&= (f_i^* \rho_{\beta(i)})_p(v, w)
\end{aligned}$$

The last identity is due to the compatibility condition of ρ, since μ is a change of orbifold charts (cf. Lemma 5.0.5). In view of Proposition 5.0.4, the compatible family $\left\{ f_i^* \rho_{\beta(i)} \middle| i \in I \right\}$ yields a unique Riemannian orbifold metric $[\hat{f}]^* \rho$.
We have to assure that $[\hat{f}]^* \rho$ does not depend on the choice of $\hat{g}$. To this end consider another representative $\hat{h} = (f, \{ h_k | k \in K \}, \mathcal{C}h_{\mathcal{V}'}, \nu') \in \operatorname{Orb}(\mathcal{V}', \mathcal{W}')$ of $[\hat{f}]$ with the same properties as $\hat{g}$. Abbreviate as $([\hat{f}]^* \rho)'$ the Riemannian orbifold metric induced by $\hat{h}$. Reviewing Proposition 5.0.4, both metrics will coincide if the family $\left\{ f_i^* \rho_{\beta(i)} \middle| i \in I \right\} \coprod \left\{ h_j^* \rho_{\beta'(j)} \middle| j \in J \right\}$ of Riemannian metrics is compatible in the sense of Lemma 5.0.5. To check this choose $i \in I$, $j \in J$ and some change of charts $\lambda \in \mathcal{C}h_{V_i,V_j'}$. Furthermore $h_j \lambda f_i^{-1}|_{f_i(\operatorname{dom}\lambda)}$ is a change of charts map. An analogous computation as above together with the compatibility of the metrics $\rho_{\beta(i)}$ and $\rho_{\beta'(j)}$ yields that λ is a Riemannian embedding. Thus $[\hat{f}]^* \rho$ and $([\hat{f}]^* \rho)'$ coincide, proving the uniqueness of the pullback orbifold metric. □

5.0.9 Remark In Lemma 5.0.8 special representatives of an orbifold diffeomorphism were used in the construction. Their lifts were given by a family of diffeomorphisms. The proof of Lemma 5.0.8 may be adapted to work with an arbitrary family of lifts of the orbifold diffeomorphism. In general these families will be families of local diffeomorphisms by Corollary 3.1.12. In this case the identities computed in the proof will only hold locally. Hence the same arguments require cumbersome notation, which may be avoided in the construction if representatives whose lifts are diffeomorphisms are used.

Our goal in introducing Riemannian orbifold metrics on orbifolds is to obtain an analogue to the Riemannian exponential map on a manifold for a Riemannian orbifold. To this end, we need to introduce the notion of a Geodesic on a Riemannian orbifold.

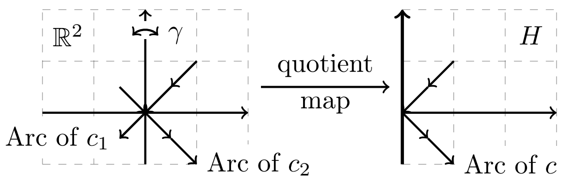
5.1. Geodesics on orbifolds

In this chapter let $(Q, \mathcal{U}, \rho)$ be a Riemannian orbifold, where the Riemannian orbifold metric is defined on the maximal atlas $\mathcal{U}$. As we introduced Riemannian orbifold metrics, the question arises, how Geodesics for a Riemannian Orbifold may be defined. Furthermore, one would like these geodesics to share at least some properties of geodesics on a Riemannian manifold. Geodesics on Riemannian orbifolds have been considered in the literature (cf. Haefliger and collaborators [12,32], Chen et al. [14])in the context of different frameworks (i.e. étale Groupoids, respectively Chen-Ruan good maps). For the setting considered in this work we shall give a definition of an Orbifold Geodesic, which shares the properties developed for geodesics on Orbifolds in the literature. In fact, the restriction of a geodesic to a compact intervall corresponds to an unique $\mathcal{G}$-geodesic in the sense of Haefliger. However since geodesics should be maps of orbifolds, our proofs are independent from this equivalence.
Throughout this section $\mathcal{I} :=]a, b[\subseteq \mathbb{R}$ will always be an open interval with $a < b$. Endow $\mathcal{I}$ with the canonical structure of an open submanifold of $\mathbb{R}$ (i.e. a trivial orbifold structure) and maximal orbifold atlas $\mathcal{U}_{\mathcal{I}}$. As a first step we define smooth paths into orbifolds:

5.1.1 Definition An orbifold map $[\hat{c}] \in \mathbf{Orb}(\mathcal{I}, (Q, \mathcal{U}))$ is called *smooth orbifold path.*

5.1.2 Example (a) If $(Q, \mathcal{U})$ is a trivial orbifold (i.e. a manifold), a smooth orbifold path is just a differentiable curve from $\mathcal{I}$ into the manifold Q.

(b) Reconsider example 2.7.2: The map $\gamma \colon \mathbb{R}^2 \to \mathbb{R}^2, (x, y) \mapsto (-x, y)$ is a reflection of $\mathbb{R}^2$ and H is the right half plane. Let $q \colon \mathbb{R}^2 \to H$ be the quotient map to the orbit space with respect to the $\langle\gamma\rangle$-action, then H is an orbifold with global chart $(\mathbb{R}^2, \langle\gamma\rangle, q)$. As the orbifold atlas contains only one chart, the change of charta are generated by γ. Define $I_1 :=]0, \frac{3}{4}[$ and $I_2 :=]\frac{1}{4}, 1[$ which cover $]0,1[= I_1 \cup I_2$. Let $\lambda \colon I_1 \supseteq I_1 \cap I_2 \to I_2$ be the canonical inclusion, then the quasi-pseudogroup $P := \left\{ \mathrm{id}_{I_1}, \mathrm{id}_{I_2}, \lambda, \lambda^{-1} \right\}$ generates the change of charts of $\{ I_1, I_2 \}$.
Consider smooth maps $c_1 \colon I_1 \to \mathbb{R}^2, t \mapsto (1-2t, 1-2t)$ and $c_2 \colon I_2 \to \mathbb{R}^2, t \mapsto (2t-1, 1-2t)$. We obtain a continuous map $c\colon]0,1[\to H, t \mapsto q \circ c_i(t),\ \forall t \in I_i$. Set $\nu(\lambda) := \gamma$, to uniquely determine $\nu \colon P \to \Psi(\mathcal{U})$, which satisfies (R4) of Definition E.2.3. Then $\hat{c} := (c, \{ c_1, c_2 \}, P, \nu)$ is a smooth path into H. We sketch the arcs of the lifts and the path:



Notice that there is the weaker notion of a continuous path. They were introduced in [12, Chapter III, 3], to obtain a fundamental group of an étale Groupoid. The map $\hat{c}$ induces a continuous path into H in the sense of Haefliger (cf. [12, III. Example 3.3 (2)]). Define a map $\nu' \colon P \to \Psi(\mathcal{U})$ via $\nu'(\lambda) = \mathrm{id}_{\mathbb{R}^2}$. The tuple $(c, \{ c_1, c_2 \}, P, \nu')$ does *not* define a charted orbifold map, but it induces a continuous path in the sense of Haefliger (cf. [12, III Example 3.3 (2)])

In the last Example, an orbifold path has been constructed with respect to a special orbifold atlas: Define the set of all orbifold charts $\mathcal{A}_{\mathcal{I}} = \{ (V_\alpha, \{ \mathrm{id}_{V_\alpha} \}, \pi_\alpha) | \alpha \in A \} \subseteq \mathcal{U}_{\mathcal{I}}$ such that an orbifold chart $(V_\alpha, \{ \mathrm{id}_{V_\alpha} \}, \pi_\alpha) \in \mathcal{U}_{\mathcal{I}}$ is contained in $\mathcal{A}_{\mathcal{I}}$ if and only if:
$V_\alpha =]l(\alpha), r(\alpha)[\subseteq \mathcal{I}$ is an open connected interval with $a \leq l(\alpha) < r(\alpha) \leq b$ and the map $\pi_\alpha \colon]l(\alpha), r(\alpha)[\to \mathcal{I}$ is the inclusion (of sets). By construction each change of orbifold charts in $\mathcal{C}h_{V_\alpha, V_\beta}$ for $(V_\alpha, \{ \mathrm{id}_{V_\alpha} \}, \pi_\alpha), (V_\beta, \{ \mathrm{id}_{V_\beta} \}, \pi_\beta) \in \mathcal{A}_{\mathcal{I}}$ is an inclusion of open sets.
Consider a smooth orbifold path $[\hat{c}] \in \mathbf{Orb}(\mathcal{I}, (Q, \mathcal{U}))$ with representative $\hat{c} := (c, \{ c_k \}_{k \in I}, [P, \nu])$, whose lifts are defined on charts $(\mathrm{dom}\, c_k, \{ \mathrm{id}_{\mathrm{dom}\, c_k} \}, \pi_k)$. The chart maps of orbifold charts on $\mathcal{I}$ are diffeomorphisms, since they are also manifold charts of the smooth manifold $\mathcal{I}$. Define an orbifold atlas $\mathcal{V}_{\hat{c}} := \{ \pi_k(\mathrm{dom}\, c_k) \}_{k \in I}$ of $\mathcal{I}$, where $\pi_k(\mathrm{dom}\, c_k) \subseteq \mathcal{I}$ is a connected open interval. Hence $\mathcal{V}_{\hat{c}} \subseteq \mathcal{A}_{\mathcal{I}}$ holds. Apply Lemma E.4.2 together with this family of charts, to obtain a representative $\hat{h} \in \mathrm{Orb}(\mathcal{V}_{\hat{c}}, \mathcal{W})$ of $[\hat{c}]$, where $\mathcal{W}$ is the range atlas of $\hat{c}$. In conclusion for each smooth orbifold path, there is a representative, whose domain atlas is contained in $\mathcal{A}_{\mathcal{I}}$. In the rest of this section we shall exploit these special representatives of a smooth orbifold path.

5.1.3 Lemma *Let $[\hat{c}] \in \mathbf{Orb}(\mathcal{I}, (Q, \mathcal{U}))$ be a smooth orbifold path and a be some point in $\mathcal{I}$. Identifying the tangent orbifold $\mathcal{T}\mathcal{I}$ with the tangent manifold $\mathcal{I} \times \mathbb{R}$, there is a unique element $\mathcal{T}c(a, 1) \in \mathcal{T}_{c(a)}Q$ called the* initial vector *of $[\hat{c}]$ at a. For each representative $\hat{c} = (c, \{ c_k \}_{k \in I}, [P, \nu]) \in \mathrm{Orb}(\mathcal{V}, \mathcal{W})$ of $[\hat{c}]$ with $\mathcal{V} \subseteq \mathcal{A}_{\mathcal{I}}$ and $a \in \mathrm{dom}\, c_k$, the initial vector is induced by $T_a c_k(1)$.*

Proof. Consider the lift $c_k \colon \mathrm{dom}\, c_k \to V_k$, where $(\mathrm{dom}\, c_k, \{ \mathrm{id}_{\mathrm{dom}\, c_k} \}, \pi_k) \in \mathcal{A}_{\mathcal{I}}$ and $(V_k, G_k, \psi_k) \in \mathcal{U}$ hold. As $\mathcal{I}$ is a trivial orbifold, the tangent manifold $T\mathcal{I} \cong \mathcal{I} \times \mathbb{R}$ coincides with the tangent orbifold. We suppress the identification $T\,\mathrm{id}_{\mathcal{I}}$ in the formulas: By Definition of the tangent orbifold map 4.1.7, $\mathcal{T}c(a, 1)$ is well-defined and $T\psi_k T c_k T(\pi_k)^{-1}(a, 1)$ holds by construction. Hence it suffices to prove $T\pi_k^{-1}(a, 1) = (a, 1) \in \mathrm{dom}\, Tc_k \cong \mathrm{dom}\, c_k \times \mathbb{R}$. As $(\mathrm{dom}\, c_k, \{ \mathrm{id}_{\mathrm{dom}\, c_k} \}, \pi_k) \in \mathcal{A}_{\mathcal{I}}$, π_k is the inclusion of sets $\mathrm{dom}\, c_k \hookrightarrow \mathcal{I}$, π_k is the restriction of a linear continuous map. A computation in the identification proves $T\pi_k^{-1}(a, 1) = (a, 1)$, whence from $T_a c_k(1) = Tc_k(a, 1)$, the assertion follows. □

Restricting orbifold paths to compact subsets yields several benefits. For instance there are representatives of orbifold maps (on a neighborhood of the compact set) with nice properties:

5.1.4 Lemma *Let $[\hat{c}] \in \mathbf{Orb}(\mathcal{I}, (Q, \mathcal{U}))$ be an orbifold path and $[a, b] \subseteq \mathcal{I}$ some compact subset. There exists a charted orbifold map $\hat{g} := (c|_{]x,y[}, \{ g_k | 1 \leq k \leq N \}, (P_g, \nu_g))$ with $x < a < b < y$ and $N \in \mathbb{N}$, such that:*

1. $[\hat{c}]|_{]x,y[} = [\hat{g}]$,
2. $\mathrm{dom}\, g_k =]l(k), r(k)[$ *for each* $1 \leq k \leq N$, *such that*

$$x = l(1) < l(2) < r(1) < l(3) < r(2) < \cdots < l(N) < r(N-1) < r(N) = y$$

3. $P_g = \left\{ \mathrm{id}_{]l(N), r(N)[} \right\} \cup \left\{ \mathrm{id}_{]l(k), r(k)[}, \iota_k^{k+1}, (\iota_k^{k+1})^{-1} \middle| 1 \leq k \leq N-1 \right\}$, *where* ι_k^{k+1} *is the canonical inclusion* $]l(k+1), r(k)[\hookrightarrow]l(k+1), r(k+1)[$

Proof. Construct a refinment of the domain atlas of $\hat{c}$. A full proof is given in Appendix F. □

5.1.5 Definition (Orbifold geodesic) Let $[\hat{c}] \in \mathbf{Orb}(\mathcal{I}, ((Q,\mathcal{U}))$ be a smooth path into a Riemannian Orbifold. The map $[\hat{c}]$ is an *orbifold geodesic*, if there there is a representative $(c, \{ c_i \}_{i\in I}, [P,\nu]) \in \mathrm{Orb}(\mathcal{V}, \{ (V_i, G_i, \psi_i \}_{i\in I})$ with $\mathcal{V} \subseteq \mathcal{A}_{\mathcal{I}}$, such that for each $i \in I$ the lift $c_i \colon]l(i), r(i)[\to V_i$ is a geodesic. Here (V_i, ρ_{V_i}) is the Riemannian manifold, where ρ_{V_i} is the member of the Riemannian orbifold metric. If $[\hat{c}]$ is a geodesic, the arc $\operatorname{Im} c \subseteq Q$ is called a *geodesic arc* .

5.1.6 Example Return to Example 2.7.2: Consider $\gamma\colon \mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto (-x,y)$ and the orbifold $\mathbb{R}^2/\langle\gamma\rangle \cong H$ (where H is the right half plane in $\mathbb{R}^2$). Endow the global chart $(\mathbb{R}^2, \langle\gamma\rangle, \psi)$ with the flat Riemannian metric. As $\langle\gamma\rangle \subseteq O(2)$, this Riemannian metric is $\langle\gamma\rangle$-invariant. Non trivial geodesics in this metric are straight lines, which induce geodesics of orbifolds. Geodesics contained either in the right or left half plane are mapped to straight lines in the quotient. Standard Riemannian geometry shows that a connected component of the set of points fixed jointly by a set of Riemannian isometries is a closed totally geodesic submanifold (cf. [40, II. Theorem 5.1]). Since $\langle\gamma\rangle$ acts by Riemannian isometries, geodesics which contain singular points, either pass through the singular locus in one point or are contained in it. Furthermore, Geodesics which pass through the singular locus, are reflected (as befits an example called mirror in $\mathbb{R}^2$). The following figure depicts an arc of this type:

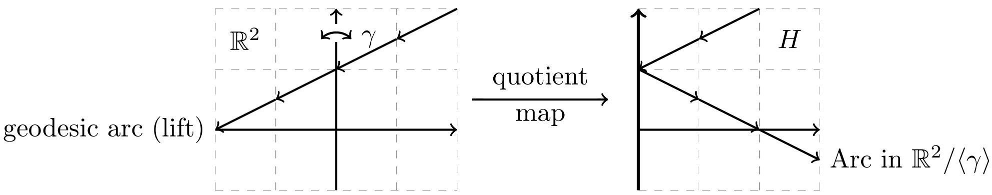


Figure 1: Orbifold geodesic in $\mathbb{R}^2/\langle\gamma\rangle$: Reflected Arc

In particular orbifold geodesics behave differently from geodesics in Riemannian manifolds. It is well known that the arc of an orbifold geodesic may be not even locally length minimizing (cf. [32, 2.4.2]). The following picture (which is slightly wrong to show the reflection) illustrates this behavior:

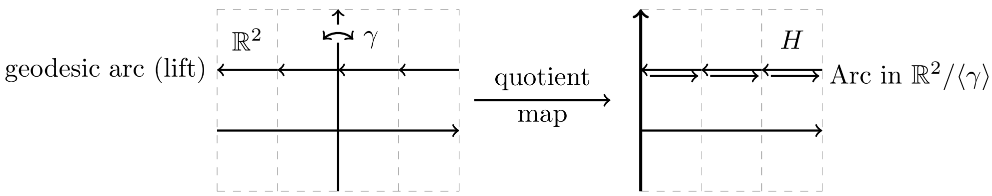


Figure 2: Orbifold geodesic in $\mathbb{R}^2/\langle\gamma\rangle$: The arc is not length minimizing in any neighborhood of the singular point.

For further examples of orbifold geodesics (in particular closed geodesics on orbifolds) we refer to [32, 2.4.5].

5.1.7 Proposition *Let $[\hat{c}] \in \mathbf{Orb}(\mathcal{I}, (Q,\mathcal{U}))$ be an orbifold geodesic together with a representative $\hat{g} = (c, \{ g_j \}_{j\in J}, [P,\nu])$ of $[\hat{c}]$. If the domain atlas of $\hat{g}$ is contained in $\mathcal{A}_{\mathcal{I}}$, each lift g_j is a geodesic.*

Proof. As $[\hat{c}]$ is an orbifold geodesic, there is a representative $\hat{c} = (c, \{ c_i \}_{i\in I}, P', \nu') \in \mathrm{Orb}(\mathcal{V}, \mathcal{V}')$ such that every c_i is a geodesic in (V_i, ρ_i). Furthermore the domain atlas of $\hat{c}$ is contained in $\mathcal{A}_{\mathcal{I}}$, $(V_i, G_i, \psi_i) \in \mathcal{V}'$ and ρ_i is the member of the Riemannian orbifold metric on this chart. Consider the lifts $g_j\colon \operatorname{dom} g_j \to W_j$ of $\hat{g}$ with respect to the charts $(W_i, H_i, \pi_i) \in \mathcal{U}$. Since $\hat{c} \sim \hat{g}$, the definition of equivalence for orbifold maps yields the following data: There are lifts ε and ε' of the identity on $\mathcal{I}$, respectively ε'' and ε''' on $(Q,\mathcal{U})$ together with a charted map of orbifolds $\hat{h}$, such that

$$\hat{c} \circ \varepsilon = \varepsilon' \circ \hat{h} \qquad\qquad \hat{g} \circ \varepsilon'' = \varepsilon''' \circ \hat{h}$$

holds. We consider for $j \in J$ some $t \in \operatorname{dom} g_j$. As $t \in \mathcal{I}$, there is some index $i \in I$ with $t \in \operatorname{dom} c_i$. Recall from Definition E.3.5 that the lifts of $\varepsilon, \varepsilon', \varepsilon''$, and ε''' are local diffeomorphism. In particular they restrict to open embeddings of orbifold charts on open sets by Proposition E.3.2. Together with Lemma E.4.4 we obtain open neighborhoods $U \subseteq \operatorname{dom} c_i$ of t and $V \subseteq V_j$ of $g_j(t)$ such that: There are change of charts morphisms $\lambda\colon \operatorname{dom} c_i \supseteq U \to \operatorname{dom} g_j$, $\mu\colon V_i \supseteq V \to W_j$, with

$$g_j \circ \lambda = \mu \circ f_i|_U. \tag{5.1.1}$$

The domain atlases are contained in $\mathcal{A}_{\mathcal{I}}$, whence $\operatorname{dom} c_i, \operatorname{dom} g_j \subseteq \mathcal{I}$ and their chart maps are induced by the inclusions of sets. Hence the change of charts $\lambda\colon U \to \operatorname{dom}\operatorname{dom} g_j$ is the inclusion of an open subset. Thus $g_j|_U = \mu \circ f_i|_U$. As $(Q,\mathcal{U},\rho)$ is a Riemannian orbifold, μ is a Riemannian isometry. Since isometries preserve geodesics (cf. [41, IV. Proposition 2.6]), the identity (5.1.1) shows that in a neighborhood of t, the map g_j is a geodesic in (W_j, ρ_j). The construction did neither depend on $j \in J$ nor on t, whence g_j is a geodesic for each $j \in J$. $\square$

Two orbifold geodesic coincide if and only if their initial vectors coincide. (cf. Lemma F.0.3). On a Riemannian manifold, geodesics are uniquely determined by their initial data in one point. The same holds for orbifold geodesics:

5.1.8 Proposition *Consider $p \in Q$, $\xi \in \mathcal{T}_pQ$.*

(a) *There is an $\varepsilon > 0$, such that there exists an orbifold geodesic $\hat{c}_\xi \in \mathrm{Orb}(]-2\varepsilon, 2\varepsilon[, (Q,\mathcal{U}))$ with initial vector ξ in 0.*
(b) *Let $[\hat{c}] \in \mathbf{Orb}(\mathcal{I}, (Q,\mathcal{U}))$ and $[\hat{c}'] \in \mathbf{Orb}(\mathcal{I}', (Q,\mathcal{U}))$ be orbifold geodesics. If there exists $a \in \mathcal{I} \cap \mathcal{I}'$, such that the initial vectors of $\hat{c}$ and $\hat{c}$ in a coincide, then the initial vectors of $[\hat{c}]$ and $[\hat{c}']$ coincide at each point in $\mathcal{I} \cap \mathcal{I}'$, whence $[\hat{c}']|_{\mathcal{I}\cap\mathcal{I}'} = [\hat{c}]|_{\mathcal{I}\cap\mathcal{I}'}$ holds.*

Proof. (a) Choose some representative $(\pi, X) \in \xi$, where $(V, G, \pi) \in \mathcal{U}$ and $X \in T_xV$ such that $T\pi(X) = \xi$. Set $x = \pi_{TV}(X)$. Let ρ_V be the member of the Riemannian orbifold metric on V, i.e. (V, ρ_V) is a Riemannian manifold. Standard Riemannian geometry (cf. [41, III. Theorem 6.4.]) shows that there is an $\varepsilon > 0$ and a geodesic $c_0\colon]-2\varepsilon, 2\varepsilon[\to V$ with initial

condition (x, X), i.e. $c_0(0) = x$ and $T_0c_0(1) = X$. Let $c := \pi \circ c_0$, $P := \left\{ \mathrm{id}_{]-2\varepsilon,2\varepsilon[} \right\}$ and $\nu\colon P \to \mathbf{\Psi}(\mathcal{U})$ be the map which sends the element of P to id_V. We obtain an orbifold geodesic $\hat{c} := (c, \{ c_0 \}, P, \nu)$. By construction the initial vector of $\hat{c}$ in 0 is ξ.

(b) Since $\mathcal{I} \cap \mathcal{I}'$ is an open submanifold of $\mathcal{I}, \mathcal{I}'$ the orbifold maps restrict to orbifold maps in $\mathbf{Orb}\,(\mathcal{I} \cap \mathcal{I}', (Q, \mathcal{U}))$. To shorten our notation we may therefore assume that $\mathcal{I} = \mathcal{I}'$ and $a = 0$ holds. Choose representatives $\hat{c} = (c, \{ c_k \}_{k \in I}, P, \nu)$ and $\hat{c}' = (c', \{ c_r \}_{r \in J}, P', \nu')$. We will check the condition of Lemma F.0.3 (b), which is equivalent to the assertion:
As a first step, show that there is $\varepsilon > 0$, such that for each $t \in]-\varepsilon, \varepsilon[$ the condition of Lemma F.0.3 (b) holds. Let c_0 be a lift of $\hat{c}$ and c_0' be a lift of $\hat{c}'$ with $0 \in \operatorname{dom} c_0 \cap \operatorname{dom} c_0'$. Set $\operatorname{cod} c_0 = V_0$ and $\operatorname{cod} c_0' = V_0'$ for orbifold charts $(V_0, \{ \mathrm{id}_{V_0} \}, \pi_0)$ respectively $(V_0', \{ \mathrm{id}_{V_0'} \}, \pi_0')$. The geodesics pass through $c(0) = c'(0)$ with initial vector $\xi \in \mathcal{T}_{c(0)}(Q, \mathcal{U})$. The construction of $\xi \in \mathcal{T}_{c(0)}(Q, \mathcal{U})$ yields a change of charts $\lambda_0\colon V_0 \supseteq U \to V \subseteq V_0'$ such that $T_0\lambda_0c_0(1) = T_0c_0'(1)$. The lifts c_0 and c_0' are geodesics and λ_0 is an isometry. Uniqueness of geodesics on Riemannian manifolds now assures that there is an $\varepsilon > 0$, such that $T_t\lambda_0c_0(1) = T_tc_0'(1)$ for all $t \in]-\varepsilon, \varepsilon[$.
Second step: The set of points $[0, \varepsilon[$, where the condition of Lemma F.0.3 (b) holds may be expanded to all of $\mathcal{I} \cap [0, \infty[$. Arguing indirectly, assume that this were not the case. To obtain a contradiction we consider the element

$$t_0 := \inf \{ t \in \mathcal{I} | t > 0,\ \nexists \lambda \in \mathcal{C}h(\mathcal{U})\ :\ T_t\lambda c_k(1) = T_tc_r'(1) \text{ for some } t \in \operatorname{dom} c_k \cap \operatorname{dom} c_r' \}$$

Let c_k be the local lift of c and c_r' be the local lift of c' such that $t_0 \in \operatorname{dom} c_k \cap \operatorname{dom} c_r'$. Their images are are contained in (V_k, G_k, π_k) respectively (V_r, G_r, π_r). The first step assures that $t_0 > 0$ and by construction, the condition of Lemma F.0.3 (b) holds for all smaller t. This forces c and c' to coincide on $[0, t_0[$ and by continuity of these maps, we obtain $c(t_0) = c'(t_0)$. Thus there is a change of charts $\lambda\colon V_k \supseteq U \to V \subseteq V_r$ with $\lambda c_k(t_0) = c_r'(t_0)$. Choose some $t < t_0$ with $c_k([t, t_0]) \subseteq \operatorname{dom} \lambda$. Since $t < t_0$ holds, there is a change of charts μ with $T_t\mu c_k(1) = T_tc_r(1)$. Shrinking the domain of μ, we may assume that μ is an open embedding of orbifold charts and $\operatorname{dom} \mu \subseteq \operatorname{dom} \lambda$ is satisfied. Now $\lambda|_{\operatorname{dom} \mu}$ is an embedding of orbifold charts mapping $\operatorname{dom} \mu$ into V_r. By Proposition 2.2.2 (d) there is an element $h \in G_r$ such that $h.\lambda|_{\operatorname{dom} \mu} = \mu$. The change of charts $\lambda_{t_0} := h.\lambda$ is a Riemannian isometry which satisfies $T_t\lambda_{t_0}c_k(1) = T_t\mu c_k(1) = T_tc_r'(1)$. We deduce that on its domain, λ_{t_0} maps the geodesic c_k to c_r. There is some $\delta > 0$, such that $c_k(]t_0 - \delta, t_0 + \delta[) \subseteq \operatorname{dom} \lambda_{t_0}$ holds. Hence $T_s\lambda_{t_0}c_k(1) = T_sc_r'(1)$ holds for each $s \in]t_0 - \delta, t_0 + \delta[$. This contradicts our choice of t_0 and thus there may be no such point in $\mathcal{I} \cap [0, \infty[$. An analogous argument for $t < 0$ shows that the condition of Lemma F.0.3 (b) holds for all of $\mathcal{I}$, whence both orbifold geodesics coincide. □

5.1.9 Lemma *Let $[\hat{c}] \in \mathbf{Orb}\,(\mathcal{I}, (Q, \mathcal{U}))$ and $[\hat{c}'] \in \mathbf{Orb}\,(\mathcal{I}', (Q, \mathcal{U}))$ be orbifold geodesics, such that for some $x_0 \in \mathcal{I} \cap \mathcal{I}'$ their initial vectors coincide. There is an unique orbifold geodesic $[\hat{c} \vee \hat{c}'] \in$* $\mathbf{Orb}\,(\mathcal{I} \cup \mathcal{I}', (Q, \mathcal{U}))$*, such that*

(a) *$[\hat{c} \vee \hat{c}']|_{\mathcal{I}'} = [\hat{c}']$ and $[\hat{c} \vee \hat{c}']|_{\mathcal{I}} = [\hat{c}]$ hold,*

(b) *let $K \subseteq \mathcal{I}$ be a compact set and $\hat{c} \in [\hat{c}]$. There is $\hat{g} \in [\hat{c} \vee \hat{c}']$ together with an open set $K \subseteq U \subseteq \mathcal{I}$, such that $\hat{g}_U$ and $\hat{c}|_U$ are equivalent as charted maps. Here $\hat{g}_U$ is the charted map, whose lifts are the lifts of $\hat{g}$ with domain contained in U and $\hat{c}|_U \in [\hat{c}]|_U$ is obtained by Lemma E.4.2 with respect to the pairs $(U \cap \operatorname{dom} c_k \hookrightarrow \operatorname{dom} c_k, \mathrm{id}_{\operatorname{cod} c_k})$, c_k is a lift of $\hat{c}$.*

Proof. It is possible to "glue together" two orbifold geodesics whose initial vector coincides in one point. This procedure, together with a full proof, can be found as Lemma F.0.4 in Appendix F. □

Standard Riemannian geometry shows, that the maximal domain $\mathcal{I}$ has to be an open subset of $\mathbb{R}$ (since the lifts of an orbifold geodesic are geodesics in suitable charts, whose maximal domain is always an open subset of $\mathbb{R}$). Naturally we have to ask whether the orbifold geodesic constructed in 5.1.8 (a) may uniquely (up to equivalence of orbifold morphisms) be extended to a maximal domain. In fact each geodesic with this initial vector in 0 may then be derived as a restriction of the maximal geodesic. The next Lemma is inspired by a Lemma due to Chen and Ruan (cf. [14, Lemma 4.2.6]):

5.1.10 Lemma *Let $p \in Q$ be any point and $\xi \in \mathcal{T}_p Q$.*

(a) *There is a unique maximal interval $\mathcal{I}_\xi$, such that an orbifold geodesic $[\hat{c}_\xi] \in \mathbf{Orb}\,(\mathcal{I}_\xi, (Q,\mathcal{U}))$ with initial vector ξ in 0 exists on $\mathcal{I}$.*
(b) *If Q is compact, then for each $\xi \in \mathcal{T}Q$, $\mathcal{I}_\xi = \mathbb{R}$ holds.*

Proof. (a) Let S_ξ be the set of all orbifold geodesics whose initial vector at 0 is ξ. Orbifold geodesics with initial vector ξ at 0 exist by Proposition 5.1.8 (a), whence S_ξ is non empty. For two elements $[\hat{c}], [\hat{c}'] \in S_\xi$ by Lemma 5.1.9 there is a join $[\hat{c} \vee \hat{c}']$ which is again an element of S_ξ. Any finite number of elements in S_ξ may be joined in this way. For $[\hat{c}] \in S_\xi$ we let $\mathcal{I}_{\hat{c}}$ be the interval such that $[\hat{c}] \in \mathbf{Orb}\,(\mathcal{I}_{\hat{c}}, (Q,\mathcal{U}))$.
Construct recursively an element $[\hat{c}_\xi] \in S_\xi$ on the open subset $\mathcal{I}_\xi := \bigcup_{[\hat{c}] \in S_\xi} \mathcal{I}_{\hat{c}}$. The set $\mathcal{I}_\xi$ is an open connected subset of $\mathbb{R}$ as a union of connected open subspaces with non-empty intersection (cf. [20, Corollary 6.1.10]). In particular it is locally compact and second countable, whence σ-compact. Choose a sequence of connected compact subsets $(C_n)_{n \in \mathbb{N}_0}$, such that $\mathcal{I}_\xi = \bigcup_{n \in \mathbb{N}_0} C_n$, $C_n \subseteq C_{n+1}^\circ$ and $0 \in C_0$ hold. By compactness of $C_n \subseteq \mathcal{I}_\xi, n \in \mathbb{N}_0$, there is a finite subset $F_n \subseteq S_\xi$, with $C_n \subseteq \bigcup_{[\hat{c}] \in F_n} \mathcal{I}_{\hat{c}}$. Joining the elements of F_n, we obtain an orbifold geodesic $[\hat{c}_n] \in S_\xi$ with $C_n \subseteq \mathcal{I}_{\hat{c}_n}$. Passing to restrictions of those orbifold geodesics by remark 3.2.4, without loss of generality $\mathcal{I}_{\hat{c}_n} \subseteq C_{n+1}^\circ$ holds for $n \in \mathbb{N}$. Recursively define reprentatives of orbifold geodesics

$$\hat{g}_0 := \hat{c}_0, \quad \hat{g}_n := \hat{g}_{n-1} \vee \hat{c}_n \text{ for } n \geq 2.$$

Adjusting our choices, by Lemma 5.1.9 (c) we may achieve: For $n \geq 3$, the restrictions $(\hat{g}_n)_{C_{n-1}^\circ}, \hat{g}_{n-1}|_{C_{n-1}^\circ}$ coincide. Thus the glueing process F.0.4 implies that after finitely many steps, lifts and the pairs $(P_{\hat{c}}, \nu_{\hat{c}})$ on C_n° do not change further. Passing to the limit, we obtain a uniquely determined maximal element $[\hat{c}_\xi] \in \mathbf{Orb}\,(\mathcal{I}_\xi, (Q,\mathcal{U}))$.

(b) Following (a) it is sufficient to prove that an orbifold geodesic $[\hat{c}] \in \mathbf{Orb}\,(\mathcal{I}, (Q,\mathcal{U}))$ with initial vector ξ at 0 and $\mathcal{I} =]a,b[$ may be extended in the following sense: If there is a sequence $(t_n)_{n \in \mathbb{N}} \subseteq]a,b[$, such that $t_n \to b$ and $\lim c(t_n)$ exists in Q, then there is an orbifold geodesic $[\hat{c}']$ defined on $]a,b'[$, $b' > b$, whose initial vector at 0 is ξ. Set $q := \lim_{n \in \mathbb{N}} c(t_n)$ and choose an orbifold chart (V, G_x, ψ) with $q = \psi(x)$ for $x \in V$. Notice that $\psi^{-1}(q) = \{\, x \,\}$ holds. Choose a compact neighborhood U_x of x and observe that $G_x.U_x$ is again a compact set. A compactness argument together with [17, 3.2 Proposition 2.5] proves that there are $\delta > 0$ and

$\varepsilon > 0$ such that for each $p \in U_x$ and $v \in B_{\rho_V}(0_q, \varepsilon)$, there is a unique geodesic $\gamma_v \colon]-\delta, \delta[\to V$ with initial value $T_0\gamma(1) = v$. Here ρ_V is the member of the Riemannian orbifold metric on V. For N large enough one obtains $c(t_n) \in \psi(U_x)$, $\forall n \geq N$. The definition of an orbifold geodesic implies that for each t_n there is some local lift $c_n \colon \operatorname{dom} c_n \to V_n$ of c with $t_n \in \operatorname{dom} c_n$ and $(V_n, H_n, \varphi_n) \in \mathcal{U}$. By compatibility of orbifold charts, $c(t_n) \in \operatorname{Im}\varphi_n \cap \psi(U_x)$ for $n \geq N$ implies that there is some change of orbifold charts λ_n with $\lambda_n c_n(t_n) \in G_x.U_x$. As each λ_n is a Riemannian embedding, the definition of an orbifold geodesic yields $\|T_{t_n}\lambda_n c_n(1)\|_{\rho_V} = K = \|T_{t_m}\lambda_m c_m(1)\|_{\rho_V}$ for each $n, m \geq N$, Using homogeneity of geodesics on Riemannian manifolds ([17, 3.2 Lemma 2.6]),for each $q \in G_x.U_x$ there is some $\delta' > 0$, such that for each $v \in B_{\rho_V}(0_q, K+1)$ the geodesic with initial value v exists up on $]-\delta', \delta'[$. Let γ_X be the geodesic in (V, ρ_V) with intial vector X. Choose $n_0 > N$ so large that $b - t_{n_0} < \delta'$ holds. The geodesic $g_{n_0} \colon]t_{n_0} - \delta, t_{n_0} + \delta'[\to V, t \mapsto \gamma_{T_{t_{n_0}}\lambda_{n_0}c_{n_0}(1)}(t - t_{n_0})$ induces an orbifold geodesic $\hat{g} := (\psi \circ g_{n_0}, \{ g_{n_0} \}), \left\{ \operatorname{id}_{]t_{n_0}-\delta', t_{n_0}+\delta'[} \right\}, \nu)$ where $\nu(\operatorname{id}_{]t_{n_0}-\delta', t_{n_0}+\delta'[}) := \operatorname{id}_V$. By construction, the initial vector of $\hat{g}$ in t_{n_0} coincides with the initial vector of $\hat{c}$ in t_{n_0}. Thus Lemma 5.1.9 yields an orbifold geodesic $\hat{c} \vee \hat{g}$ which is defined on $]a, t_{n_0} + \delta'[$. The initial vector of $\hat{c} \vee \hat{g}$ in 0 is ξ and its domain strictly contains $]a, b[$.

□

5.1.11 Remark The maximal geodesics $[\hat{c}_\xi]$ on $\mathcal{I}_\xi$ constructed in Lemma 5.1.10 (a) do not extended, i.e. if $[\hat{g}] \in \mathbf{Orb}(\mathcal{I}, (Q, \mathcal{U}))$ is a geodesic whose initial vector at $a \in \mathcal{I} \cap \mathcal{I}_\xi$ coincides with the initial vector of $[\hat{c}_\xi]$ in a, then $\mathcal{I} \subseteq \mathcal{I}_\xi$ and $[\hat{c}_\xi]|_{\mathcal{I}} = [\hat{g}]$ hold.

5.1.12 Theorem *Let $(Q, \mathcal{U}, \rho)$ be a Riemannian orbifold and $\xi \in \mathcal{T}Q$.*

(a) *There exist an open neighborhood $O_\xi \subseteq \mathcal{T}Q$ of ξ and $\delta, \delta' > 0$ such that there is a continuous map $\alpha_\xi \colon]-\delta, \delta'[\, \times O_\xi \to Q$ and for $\xi' \in O_\xi$ the path $\alpha_{\xi'} \colon]-\delta, \delta'[\, \to Q, t \mapsto \alpha(t, \xi')$ is th geodesic arc of an orbifold geodesic $[\hat{c}_{\xi'}]$ with initial vector ξ' in 0. We call α_ξ an* orbifold geodesic flow .

(b) *Consider a pair $(\xi, \zeta) \in \mathcal{T}Q \times \mathcal{T}Q$ with $O_\xi \cap O_\zeta \neq \emptyset$, then α_ξ and α_ζ coincide on the intersection of their respective domains.*

(c) *If the maximal orbifold geodesic $[\hat{c}_\xi]$ with initial vector ξ in 0 satisfies $[c, d] \subseteq \mathcal{I}_\xi$, the set O_ξ in (a) may be constructed, such that for $\zeta \in O_\xi$ the orbifold geodesic $[\hat{c}_\zeta]$ is defined on $[c, d]$.*

Proof. (a) There is some $\varepsilon > 0$ together with the representative of an orbifold geodesic $\hat{c} = (c, \{ g_i | 1 \leq i \leq N \}, P, \nu)$ defined on $]-2\varepsilon, 2\varepsilon[$ with initial vector ξ in 0 by Proposition 5.1.8 (a). Shrinking the domain, without loss of generality $\hat{c}$ is defined on an open neighborhood of $[-\varepsilon, \varepsilon]$ with properties as in Lemma 5.1.4. We claim that there is an open neighborhood of ξ, such that each orbifold geodesic with initial vector in this set, exists at least on $[0, \varepsilon]$.

Proof of the claim: To shorten the notation relable the charts as $\{ -t, -t+1, \ldots, 0, 1, \ldots, s \}$ for certain $s, t \in \mathbb{N}_0$, such that $0 \in \operatorname{dom} g_0$. Let $g_i \colon]l(i), r(i)[\to U_i$, $-t \leq i \leq s$ be the lifts, where the (U_i, G_i, ψ_i) are charts in $\mathcal{U}$. By construction for $-t \leq i < s$ there is a change of charts λ_i^{i+1} satisfying $\lambda_i^{i+1} g_i|_{]l(i+1), r(i)[} = g_{i+1}|_{]l(i+1), r(i)[}$. Choose for $1 < i \leq s$ a point

$z_i \in]l(i), r(i-1)[$, with $z_0 := 0 < z_i < z_j$ for $i < j$. Define $X_i := T_{z_i} g_i(1))$ for $1 \leq i \leq s$ and observe that g_i is uniquely determined by X_i. By construction $(\psi_0, X_0) \in \xi$ holds. Finally choose $z_{s+1} \in \operatorname{dom} g_s$ with $z_{s+1} > \varepsilon$.

Standard Riemannian geometry on manifolds shows that the geodesic flow depends smoothly on the initial data (cf. [17, 3.2 Proposition 2.5], resp. [43, IV, §3 and VII, §7]). On the Riemannian manifold (U_i, ρ_i) there is a geodesic flow $\varphi_i \colon \mathcal{D}_i \to TU_i$, defined on an open set $\mathcal{D}_i \subseteq \mathbb{R} \times TU_i$ (cf. [43, IV, §4 remark before Corollary 4.3]). The map φ_i is smooth by an application of [43, IV, §2 Thm 2.6]. Since g_s is a geodesic defined on $[z_s, z_{s+1}] \subseteq]l(s), r(s)[$ with $T_{z_s} g_s(1) = X_s$, the compact set $[z_s, z_{s+1}] \times \{ X_s \}$ is contained in the open set $\mathcal{D}_N$. An application of Wallace Theorem [20, 3.2.10] provides an open neighborhood $[z_s, z_{s+1}] \times \{ X_s \} \subseteq]z_s - \delta_s, z_{s+1} + \delta_s[\times V_s \subseteq \mathcal{D}_s$. For each element ζ of this neighborhood in TU_s, the geodesic with initial data ζ exists on the interval $]z_s - \delta_s, z_s + \delta_s[$.

Shrinking V_s and δ_s, we may assume, that $V_s \subseteq \pi_{TU_s}^{-1}(\operatorname{cod} \lambda_{s-1}^s)$ and $z_s - \delta_s > r(s-2)$ hold. Identify $T \operatorname{cod} \lambda_{s-1}^s$ and $T \operatorname{dom} \lambda_{s-1}^s$ with open subsets of TU_s respectively TU_{s-1} and set $V_s' := (T\lambda_{s-1}^s)^{-1}(V_s) \subseteq TU_{s-1}$. The geodesic g_{s-1} is determined by X_{s-1} and its domain $]l(s-1), r(s-1)[$ contains $[z_{s-1}, z_s]$ with $T_{z_s} g_{s-1}(1) \in V_s'$. As the geodesic flow φ_{s-1} is smooth, arguments as above applied to φ_{s-1} yield an open set $V_{s-1} \subseteq TU_{s-1}$ with:

- $[z_{s-1}, z_s] \times \{ X_{s-1} \} \subseteq]z_{s-1} - \delta_{s-1}, z_s + \delta_{s-1}[\times V_{s-1} \subseteq T(\operatorname{cod} \lambda_{s-2}^{s-1})$,
- $V_{s-1} \subseteq \lambda_{s-1}^{-1}(V_s')$,
- $z_{s-1} - \delta_{s-1} > r(s-3)$.

Again one obtains an open set $V_{s-1}' := (T\lambda_{s-2}^{s-1})^{-1}(V_{s-1}) \subseteq TU_{s-2}$. Repeating the argument for each $0 \leq i \leq s-2$, we derive an open neighborhood $V_0 \subseteq TU_0$ of X_0. For each $\zeta \in V_0$, there is a unique family of geodesics $\left\{ c_\zeta^i \middle| 0 \leq i \leq s \right\}$, such that c_ζ^i is defined at least on $]z_i - \delta_i, z_{i+1} + \delta_i[$. In addition these families satisfy $T_{z_i} \lambda_{i-1}^i c_{i-1}(1) = T_{z_i} c_i(1)$.

Repeating the argument for $[-\varepsilon, 0]$, we obtain an open set V_0^-, such that the the geodesics are defined onf $[-\varepsilon, 0]$. Set $V := V_0 \cap V_0^-$ and $\delta := z_{-t-1} - \delta_{-t}$ and $\delta' := z_{s+1} + \delta_s$. For each $\zeta \in V$ and $-t \leq i \leq s+1$, the geodesics c_ζ^i are defined on $[z_{i-1} - \delta_i, z_i + \delta_i]$. By construction one may restrict their domains, such that $\lambda_i^{i+1} c_\zeta^i|_{]z_{i+1} - \delta_{i+1}, z_{i+1} + \delta_i[} = c_\zeta^{i+1}|_{]z_{i+1} - \delta_{i+1}, z_{i+1} + \delta_i[}$ holds. For each $\zeta \in V$, the family $\left\{ c_\zeta^i \right\}_{-t \leq i \leq s}$ induces an orbifold geodesic. The continuity of the geodesic flows yields a well-defined continuous map

$$\tilde{\alpha} \colon]-\delta, \delta'[\times V \to Q, (t, \zeta) \mapsto \psi_i(c_\zeta^i(t)) \text{ for each } t \in]z_i - \delta_i, z_{i+1} + \delta_i[.$$

Consider the orbifold chart $(TU_0, G_0, T\psi_0) \in \mathcal{TU}$ for the tangent orbifold $\mathcal{T}(Q, \mathcal{U})$. Chart maps of orbifold charts are open maps and thus $O_\xi := T\psi_0(V)$ is open in $\mathcal{T}Q$. It contains $\xi = T\psi_0(X_0)$ and the subspace topology on O_ξ with respect to Q coincides with the quotient topology induced on O_ξ by $T\psi_0$ (since $T\psi_0$ factors via a homeomorphism with open image). The restriction $q := T\psi_0|_V^{O_\xi}$ is an open, continuous and surjective map. For each $\zeta \in O_\xi$, choose a preimage $\tilde{\zeta} \in q^{-1}(\{ \zeta \}) \in V$. Notice that each choice of preimage for ζ induces an orbifold geodesic with initial vector ζ at 0. Following Proposition 5.1.8 (b) the geodesic arcs obtained from a choice of $q^{-1}(\zeta)$ coincide with the arc of $[\hat{c}_\zeta]$ on the intersection of their domains. Hence each choice defines the same continuous path into Q. As $\hat{c}_\zeta$ is defined at least on $]\delta, \delta'[$ the maximal geodesic with initial vector ζ is defined on this interval. We derive a

well-defined map

$$\alpha\colon]-\delta,\delta'[\times O_p \to Q,\ (t,\zeta)\mapsto \tilde{\alpha}(t,\tilde{\zeta})$$

The map $\mathrm{id}_{]-\delta,\delta'[}\times q$ is the product of open continuous surjective maps, whence it is itself open, continuous and surjective. In particular this mapping is a quotient map such that $\tilde{\alpha} = \alpha \circ (\mathrm{id}_{]-\delta,\delta'[}\times q)$ holds. As $\tilde{\alpha}$ is continuous, [19, VI. Theorem 3.1] implies that α is a continuous map.

(b) By Proposition 5.1.8 (b) the arcs of two orbifold geodesics with the same initial data coincide. Hence for each $\omega \in O_\xi \cap O_\zeta$, the arcs of the geodesics coincide, therefore $\alpha_\xi(\cdot,\omega)$ and $\alpha_\zeta(\cdot,\omega)$ coincide on the intersection of their respective domains. This proves the assertion.

(c) Repeat the proof of (a) verbatim with $[c,d]\subseteq \mathcal{I}$ instead of $[-\varepsilon,\varepsilon]$.

□

5.1.13 Corollary *For every $p\in Q$, there is an open neighborhood $W_p\subseteq \mathcal{T}Q$ of $0\in \mathcal{T}_pQ$, such that there is a continuous map $\alpha\colon]-2,2[\times W_p\to Q$ and $t\mapsto \alpha(t,\xi),\ t\in]-2,2[$ is the unique geodesic arc with initial vector ξ in 0 defined on $]-2,2[$.*

Proof. Choose an arbitrary orbifold chart (U,G,ψ) such that $p=\psi(x)$ for some $x\in U$. By definition $T\psi(0_x)=0_p\in \mathcal{T}_pQ$ holds, where $0_x\in T_xU$ is the zero element. Standard Riemannian geometry (see [17, 3.2 Proposition 2.7]) assures that there is a smooth mapping $\gamma\colon]-2,2[\times V\to U$, defined on some open set $V\subseteq TU$, such that each $x\in V$ induces a geodesic in U defined at least on $]-2,2[$. Arguing as in the proof of theorem 5.1.12, we choose $W_p := T\psi(V)$ and $\alpha\colon]-2,2[\times W_p\to Q,(t,\xi)\mapsto \gamma(t,x_\xi)$, where x_ξ is an arbitrary preimage of ξ under $T\psi$ in V. □

Albeit the quite similar behavior of orbifold geodesics to geodesics on Riemannian manifolds, not all properties of geodesics may be preserved in the orbifold case. For example as is noted in [32, 2.4.2] orbifold geodesics may not even be locally length minimizing in the natural length metric on Q (induced by piecewise differentiable paths). However as we are only interested in geodesics as a tool to obtain an exponential map we shall not investigate this behavior.

5.2. The Riemannian orbifold exponential map

In this section our main tool derived from Riemannian geometry on orbifolds, the Riemannian orbifold exponential map, is introduced. Again the triple $(Q, \mathcal{U}, \rho)$, will be a Riemannian orbifold, where the Riemannian orbifold metric ρ is defined on the maximal atlas $\mathcal{U}$ (cf. Proposition 5.0.4). By Lemma 5.1.10 (a) for each $\xi \in \mathcal{T}Q$, there is a maximal orbifold geodesic $[\hat{c}_\xi]$ with initial vector ξ in 0. The geodesic arc of a maximal orbifold geodesic is unique by Proposition 5.1.8. Hence the continuous map of the base spaces $c_\xi \colon \mathcal{I}_\xi \to Q$ is uniquely determined.

5.2.1 Definition (Riemannian orbifold exponential map) Let Ω be the set of all $\xi \in \mathcal{T}Q$, such that the orbifold geodesic $[\hat{c}_\xi]$ with underlying map $c_\xi \colon \mathcal{I}_\xi \to Q$ satisfies $[0,1] \subseteq \mathcal{I}_\xi$. The map

$$\exp_{\mathrm{Orb}} \colon \Omega \to Q, \ \xi \mapsto c_\xi(1)$$

is called *Riemannian orbifold exponential map*. The set Ω is an open neighborhood of the zero section, by Theorem 5.1.12 (c) and Corollary 5.1.13. We call Ω *domain of the Riemannian orbifold exponential map.*

5.2.2 Lemma *The Riemannian orbifold exponential map is continuous and for each $0_p \in \mathcal{T}_pQ$ the identity $\exp_{\mathrm{Orb}}(0_p) = p$ holds.*

Proof. Let $\xi \in \Omega$ be arbitrary. The geodesic $[\hat{c}_\xi]$ is defined on an open intervall $\mathcal{I}_\xi$, such that $[0,1] \subseteq \mathcal{I}_\xi$ holds. By Theorem 5.1.12 (c), there is an open neighborhood $\xi \in O_\xi \subseteq \mathcal{T}Q$, such that each orbifold geodesic $[\hat{c}_\omega]$ for $\omega \in O_\xi$ is defined on $[0,1] \subseteq]-\delta, \delta'[$. Furthermore $O_\xi \subseteq \Omega$ holds. There is a continuous map $\alpha_\zeta \colon]-\delta, \delta'[\times O_\xi \to Q, (t, \omega) \mapsto \hat{c}_\omega(t)$, such that by construction $\exp_{\mathrm{Orb}}(\omega) = \alpha_\xi(1, \omega)$, $\forall \omega \in O_\xi$ is satisfied. Hence $\exp_{\mathrm{Orb}}$ restricts to a continuous map on the open set O_ξ. Theorem 5.1.12 (b) assures that for any $\zeta \in \Omega$ the maps $\alpha_\zeta(1, \cdot)$ and $\alpha_\xi(1, \cdot)$ coincide on $O_\xi \cap O_\zeta$. From [19, IV. Theorem 9.4] we deduce that $\exp_{\mathrm{Orb}}$ is continuous.
Choose an arbitrary orbifold chart $(U, G, \psi) \in \mathcal{U}$, such that $p \in \psi(x)$ for some $x \in U$. The chart $T\psi$ maps $0_x \in T_xU$ to $0_p \in \mathcal{T}_pQ$. Standard Riemannian geometry assures, that the geodesic γ starting in x with velocity 0 is constant and hence defined on all of $\mathbb{R}$. Setting $c \colon \mathbb{R} \to Q, t \mapsto p$, we obtain a representative of an orbifold geodesic $\hat{c} := (c, \gamma, \{\mathrm{id}_\mathbb{R}\}, \nu)$, where $\nu(\mathrm{id}_\mathbb{R}) := \mathrm{id}_U$. The orbifold geodesic $[\hat{c}]$ has initial vector 0_p in 0 and its arc is uniquely determined by Proposition 5.1.8. This proves $\exp_{\mathrm{Orb}}(0_p) = p$. □

5.2.3 Proposition *Consider the open suborbifold $(\Omega, \mathcal{U}_\Omega)$. The map $\exp_{\mathrm{Orb}}$ induces a map of orbifolds $[\exp_{\mathrm{Orb}}] \in \mathbf{Orb}((\Omega, \mathcal{U}_\Omega), (Q, \mathcal{U}))$ also called* Riemannian orbifold exponential map.

Proof. The subset $\Omega \subseteq \mathcal{T}Q$ is open and we endow it with an orbifold atlas

$$\mathcal{T}\mathcal{U}_\Omega := \{ (U, G, \psi) \in B_{\mathcal{T}\mathcal{U}} | \psi(U) \subseteq \Omega \}$$

induced by $B_{\mathcal{T}\mathcal{U}}$. Following Remark 3.2.4, $(\Omega, \mathcal{T}\mathcal{U}_\Omega)$ is an open suborbifold of $(\mathcal{T}Q, \mathcal{B}_{\mathcal{T}\mathcal{U}})$. We claim that there is a representative $\mathcal{V}$ of $\mathcal{T}\mathcal{U}_\Omega$, together with a family of lifts, turning $\exp_{\mathrm{Orb}}$ into a

charted orbifold map $\mathrm{Orb}(\mathcal{V},\mathcal{W})$ for some $\mathcal{W}\subseteq\mathcal{U}$. By Lemma 5.2.2 the map $\exp_{\mathrm{Orb}}$ is continuous. Construct smooth lifts of $\exp_{\mathrm{Orb}}$: To this end consider arbitrary $\xi\in\Omega$. By Theorem 5.1.12 there is an open neighborhood $\xi\in O_\xi\subseteq\Omega$ together with the following data:

- $(TU_1,G_1,T\psi_1)\in T\mathcal{U}$, with $O_\xi=T\psi_1(V)\subseteq T\psi_1(TU_1)$ for some open $V\subseteq TU_1$,
- a family of orbifold charts $\{\,(U_i,G_i,\psi_i)|1\leq i\leq N\,\}\subseteq\mathcal{U}$,
- there exists a continuous map $\theta\colon V\to Q, X\mapsto\tilde{\alpha}(1,X)$, such that $\theta=\exp_{\mathrm{Orb}}\circ T\psi_{1|V_1}$ holds. The map θ is the composition of the geodesic flows φ_i on $(U_i,\rho_i), 1\leq i\leq N$, change of charts morphisms λ_{ii+1} for $1\leq i<N$, the bundle projection of TU_N and the orbifold chart ψ_N.

Recall from the proof of theorem 5.1.12, that there is a partition $0=t_0<t_1<\cdots<t_N<1$ such that a smooth map $\mathrm{Exp}_\xi\colon TU_1\supseteq V\to U_N$ may be defined via

$$\mathrm{Exp}_\xi(X):=\pi_{TU_N}\varphi_N(1-t_N,\cdot)\circ T\lambda_{N-1N}\circ\varphi_{N-1}(t_N-t_{N-1},\cdot)\circ\cdots\circ T\lambda_{12}\circ\varphi_1(t_1,\cdot)(X). \quad (5.2.1)$$

Reviewing Theorem 5.1.12, the identity $\theta=\psi_N\circ\mathrm{Exp}_\xi$ holds.
Choose an open G_1-stable subset W of V which contains some preimage x_ξ of ξ. Restricting Exp_ξ to W, we obtain a smooth map Exp_W on an orbifold chart $(W,G_W,T\psi_1|_W)$. By construction Exp_W is a smooth lift of $\exp_{\mathrm{Orb}}$ on W.
We claim that any local lift Exp'_W of $\exp_{\mathrm{Orb}}$ with respect to the charts $(W,G_W,T\psi_1|_W)$ and (U_N,G_N,ψ_N) obtained in this way, coincides with $\gamma.\mathrm{Exp}_W$ for some $\gamma\in G_N$.
The lifts Exp_W and Exp'_W are defined as restriction of a composition of geodesic flows φ_i, change of chart maps λ_{kk+1} and the bundle projection π_{TU_N} (cf. (5.2.1)). Each $\varphi_i(t_i-t_{i-1},\cdot)$ is defined on an open subset of TU_i. It is a diffeomorphism from this subset onto its (open) image in TU_i (this follows from [43, IV, § 2 , Thm. 2.9.]). The change of chart maps $T\lambda_{kk+1}$ are smooth embeddings with open images. In addition the bundle projection π_{TU_N} is an open map, whence Exp_W is an open map as a composition of such maps. The same holds for Exp'_W, whose image is contained in (U_N,G_N,ψ_N). The construction of the lifts Exp_W and Exp'_W shows that there are diffeomorphisms $\phi_W\colon W\to O$, $\phi'_W\colon W\to O'$ onto open sets $O,O'\subseteq TU_N$ with $\mathrm{Exp}_W=\pi_{TU_N}\circ\varphi_N(1-t_N,\cdot)\circ\phi_W$ and $\mathrm{Exp}'_W=\pi_{TU_N}\circ\varphi_N(1-t'_N,\cdot)\circ\phi'_W$. Without loss of generality, taking the maximum of t_N,t'_N, we may assume $t_N=t'_N$. Observe that we obtain a diffeomorpism $\phi_W\circ\phi'^{-1}_W\colon O'\to O$. For each $X\in O'$, there are unique geodesics $\gamma'_X(t):=\pi_{TU_N}\varphi_N(t,X)\colon[0,1-t_N]\to U_N$ and $\gamma_X(t):=\pi_{TU_N}\varphi_N(t,\phi_W\circ\phi'^{-1}_W(X))\colon[0,1-t_N]\to U_N$. The geodesics γ_X,γ'_X lift the same orbifold geodesic arc, since Exp_W and Exp'_W are restrictions of orbifold geodesic flows. By Lemma F.0.3 for $X\in O'$ there is some $g_X\in G_N$ with $T_{1-t_N}g_X.\gamma_X(1-t_N)=T_{1-t_N}\gamma'_X(1-t_N)$.
The element g_X acts as Riemannian isometry, mapping geodesics into geodesics, which implies $g_X.\gamma_X(t)=\gamma'_X(t)$, $\forall t\in[0,1-t_N]$. For any non-singular $X\in O'$ the isometry g_X is uniquely determined: To prove this, let $g'_X\in G_N$ be another isometry with $g'_X.\gamma_X=\gamma'_X$. Summing up this yields

$$Tg_X(X)=Tg_X.\varphi_N(0,X)=\varphi_N(0,\phi_W(\phi'_W)^{-1}(X))=Tg'_X.\varphi_N(0,X)=Tg'_X(X).$$

Since X is non singular, $T_{\pi_{TU_N}(X)}g_X=T_{\pi_{TU_N}}g'_X$ and by [48, Lemma 2.10] $g_X=g'_X$ follows. The set $O'\subseteq TU_N$ is an open, connected set. Hence Lemma B.2.3 implies that $C:=O'\setminus\Sigma_{TG_N}$ is connected. As we have seen, for each $X\in C$, there is a unique g_X with $g_X.\gamma_X(0)=\gamma'_X(0)$. The set $H_{g_X}:=\{\,c\in C|g_X.\gamma_c(0)=\gamma'_c(0)\,\}=\{\,c\in C\big|g_X.\pi_{TU_N}\varphi_N(1-t_N,c)=\pi_{TU_N}\varphi_N(1-t_N,\phi'_W\circ\phi_W^{-1}\,\}$

is a closed set by [20, Theorem 1.5.4]. Uniqueness of g_X proves that two such sets H_g and H_h are disjoint if and only if $g \neq h$ holds. Since G_N is finite, the set H_{g_X} is open and closed. By connectedness of $O' \setminus \Sigma_{TG_W}$, there is a unique $\gamma \in G_N$, with

$$\gamma.\pi_{TU_N}\varphi_N(1-t_N,\cdot)_{|O'\setminus\Sigma_{TG_N}} = \pi_{TU_N}\varphi_N(1-t_N,\cdot)\circ\phi_W\phi'^{-1}_{W|O'\setminus\Sigma_{TG_N}}. \tag{5.2.2}$$

The set $O' \setminus \Sigma_{G_N}$ is dense in O' by Newmans Theorem B.2.1. Hence by continuity (5.2.2) holds on all of O'. As $(\phi'_W)^{-1}(O') = W$ holds by cosntruction, we finally derive $\gamma.\mathrm{Exp}_W = \mathrm{Exp}'_W$.

The construction of lifts did not depend on ξ, thus we may cover Ω with a family of orbifold charts $\mathcal{V} := \{\, (W_i, G_i, \pi_i) | i \in I \,\}$, such that on each (W_i, G_i, π_i) there exists a local lift Exp_{W_i} of $\exp_{\mathrm{Orb}}$ with respect to (W_i, G_i, π_i) and a suitable chart (U_i, G_i, ψ_i). Shrinking those charts if necessary, without loss of generality we may assume $(W_i, G_i, \pi_i) \neq (W_j, G_j, \pi_j), (U_i, G_i, \psi_i) \neq (U_j, G_j, \psi_j)$ for $i \neq j$. The charts in $\mathcal{V}$ are compatible since they are contained in $\mathcal{B}_{\mathcal{TU}}$, their images cover Ω and we have $\mathcal{V} \subseteq \mathcal{TU}_\Omega$.
We claim that it is possible to construct a quasi-pseudogroup P and a map ν, such that the lifts commute with the change of charts morphisms as in defintion E.2.3. To this end consider arbitrary local lifts Exp_W respectively Exp'_W of $\exp_{\mathrm{Orb}}$ with respect to the charts (W, G, π), (U, H, ψ) respectively. (W', G', π'), (U', H', ψ'). Furthermore let $h \in \mathcal{C}h_{\mathcal{V}}$ be a change of charts morphism which induces a commutative diagramm:

$$\begin{tikzcd}
\operatorname{dom} h \arrow[r, "\mathrm{inc}"] \arrow[dd, "h"] & W \arrow[rrr, "\mathrm{Exp}_W"] \arrow[dr, "\pi"] & & & U \arrow[dl, "\psi"'] \\
& & \Omega \arrow[r, "\exp_{\mathrm{Orb}}"] & Q & \\
\operatorname{cod} h \arrow[r, "\mathrm{inc}"] & W' \arrow[rrr, "\mathrm{Exp}_{W'}"] \arrow[ur, "\pi'"] & & & U' \arrow[ul, "\psi'"]
\end{tikzcd} \tag{5.2.3}$$

Cover $\mathrm{Exp}_W(\operatorname{dom} h)$ with the domains of suitable change of charts morphisms. Our goal is to restrict h to open subsets, such that there are change of charts which complement the right hand side of (5.2.3) to a commuting triangle. By commutativity of (5.2.3) for each $X \in \operatorname{dom} h$, there is an embedding of orbifold charts $\lambda_X \in \mathcal{C}h(U, U')$, such that $\lambda_X(\mathrm{Exp}_W(X)) = \mathrm{Exp}_{W'}(h(X))$ holds. Choose a G'-stable open $\mathrm{Exp}_W(X)$−neighborhood $S \subseteq \operatorname{dom} \lambda_X$ with $G'_{\mathrm{Exp}_W(X)} = G'_S$. Applying [17, Ch. 3, Prop. 4.2], we may choose a geodesic ball $B_\beta(\mathrm{Exp}_W(X)) \subseteq S$ which is strongly convex. The isotropy group $G_{\mathrm{Exp}_W(X)}$ acts via Riemannian isometries of U, whence it commutes with the Riemannian exponential map on U. Thus we obtain $G_{\mathrm{Exp}_W(X)}.B_\beta(\mathrm{Exp}_W(X)) = B_\beta(\mathrm{Exp}_W(X))$. Restricting λ_X to $B_\beta(\mathrm{Exp}_W(X))$, without loss of generality $\operatorname{dom} \lambda_X$ is strongly convex.
Again let $\phi_W, \phi_{W'}$ denote the diffeomorphisms with $\mathrm{Exp}_W = \pi_{TU}\varphi_U(1-t_N,\cdot)\circ\phi_W$ and $\mathrm{Exp}'_W = \pi_{TU'}\varphi'_U(1-t'_N,\cdot)\circ\phi_{W'}$. Strong convexity of $\operatorname{dom}\lambda_X$ assures that there is some $\varepsilon > t_N, t'_N$, such that $\pi_{TU}\varphi_U(1-t, \varphi_U(\varepsilon - t_N, \phi_W(X)) \in \operatorname{dom}\lambda_X$ holds for all $t \in [0, 1-\varepsilon]$. Define for $Y \in W$ the element $\tilde{Y} := \varphi_U(\varepsilon - t_N, \phi_W(Y)) \in TU$. The geodesic flow is continuous and the open set $\varphi_U^{-1}(T \operatorname{dom}\lambda_X)$ contains $[0, 1-\varepsilon] \times \left\{ \tilde{X} \right\}$. Wallace Theorem [20, 3.2.10] assures that there is an open neighborhood $\tilde{X} \in \tilde{V} \subseteq TU$ such that $[0, 1-\varepsilon] \times \tilde{V} \subseteq \varphi_U^{-1}(T \operatorname{dom}\lambda_X)$ holds. Choose an open G-stable subset

X-neighborhood $V \subseteq (\varphi_U(\varepsilon - t_N, \cdot) \circ \phi_W)^{-1}(\tilde{V}) \cap \operatorname{dom} h$ with $G_V = G_X$. For each $\tilde{Y}$ with $Y \in V$, the geodesic $\gamma_{\tilde{Y}}(t) := \pi_{TU}\varphi_U(t, \tilde{Y}), t \in [0, 1-\varepsilon]$ is contained in $\operatorname{dom} \lambda_X$. We obtain two local lifts $\operatorname{Exp}'_{W|h(V)}$ and $\lambda_X \circ \operatorname{Exp}_W \circ h^{-1}_{|h(V)}$ with respect to the charts $(h(V), G'_{h(V)}, \pi'|_{h(V)})$ and (U', G', ψ'). The map λ_X is a Riemannian embedding into U' and thus commutes with parallel displacement (see [41, IV. Prop. 2.6]) of the open submanifold $\operatorname{dom} \lambda_X$. Hence we derive $T\lambda_X \varphi_U(1-\varepsilon, \tilde{Y}) = \varphi_{U'}(1-\varepsilon, T\lambda_X(\tilde{Y}))$ for $\tilde{Y} \in \tilde{V}$. In particular the following identity holds:

$$\begin{aligned} \lambda_X \circ \operatorname{Exp}_W \circ h^{-1}|_{h(V)} &= \pi_{TU'} T\lambda_X \varphi_U(1-\varepsilon, \cdot)\varphi_U(\varepsilon - t_N, \cdot) \circ \phi_W \circ h^{-1}|_{h(V)} \\ &= \pi_{TU'} \varphi_{U'}(1-\varepsilon, \cdot) T\lambda_X \varphi_U(\varepsilon - t_N, \cdot) \circ \phi_W \circ h^{-1}|_{h(V)} \end{aligned} \tag{5.2.4}$$

The local lifts $\lambda_X \operatorname{Exp}_W h^{-1}|_{h(V)}$ and $\operatorname{Exp}_W|'_{h(V)}$ are therefore compositions of the bundle projection $\pi_{TU'}$, the geodesic flow on U' and some diffeomorphism. As we have already seen, there is some $\gamma \in H'$, such that $\gamma.\lambda_X \operatorname{Exp}_W h^{-1}_{|h(V)} = \operatorname{Exp}'_{W|h(V)}$ holds. Replacing λ_X with the embedding of orbifolds $\gamma.\lambda_X$ we derive

$$\lambda_X \operatorname{Exp}_W|_V = \operatorname{Exp}_{W'} \circ h|_V. \tag{5.2.5}$$

We may thus cover $\operatorname{dom} h$ by a family of open G-stable subsets $\{ W_{X_i} | i \in I_h \}$, such that for each $h_i := h_{|W_{X_i}}$, there is a change of charts morphism λ_i^h which satisfies equation (5.2.5). Repeating this construction for every change of charts in $\mathcal{C}h_{\mathcal{V}}$, we obtain a family $P := \{ h_i | i \in I_h,\ h \in \mathcal{C}h_{\mathcal{V}} \}$. By cosntruction P is a quasi-pseudogroup, which generates $\mathbf{\Psi}(\mathcal{V})$. For each element f of P choose and fix some $h \in \mathcal{C}h_{\mathcal{V}}$ with $f = h_i$ and define the map $\nu \colon P \to \mathbf{\Psi}(\mathcal{U}), f = h_i \mapsto \lambda_i^h$. By construction $\widehat{\exp_{\mathrm{Orb}}} := (\exp_{\mathrm{Orb}}, \{ \operatorname{Exp}_W | (W, G, \pi) \in \mathcal{V} \}, P, \nu)$ satisfies conditions (R1)-(R4a) of definition E.2.3. We check condition (R4b), i.e. given $g, h \in P$ and $x \in \operatorname{dom} h \cap \operatorname{dom} g$ with $\operatorname{dom} g, \operatorname{dom} h \subseteq U$ and $\operatorname{germ}_x h = \operatorname{germ}_x g$, then $\operatorname{germ}_{\operatorname{Exp}_U(x)} \nu(h) = \operatorname{germ}_{\operatorname{Exp}_U(x)} \nu(g)$ holds.
Let $\operatorname{dom} \nu(h) \subseteq V$ and $\operatorname{cod} \nu(h) \subseteq V'$, where $(V, H, \psi), (V', H', \psi')$ are suitable orbifold charts. By construction we already know $\nu(h)(\operatorname{Exp}_U(x)) = \nu(g)(\operatorname{Exp}_U(x))$. Restricting to an open and $H_{\operatorname{Exp}_U(x)}$-stable subset $x \in S_x$ of $\operatorname{dom} \nu(g) \cap \operatorname{dom} \nu(h)$, the change of charts $\nu(g)$ and $\nu(h)$ restrict to open embeddings of orbifold charts. By Proposition 2.2.2 there is a unique $\gamma \in H'$ such that $\gamma.\nu(g)_{|S_x} = \nu(h)_{|S_x}$. Now $\gamma.\nu(g)(x) = \nu(h)(x) = \nu(g)(x)$ implies that $\gamma \in H'_{\nu(g)(S_x)}$ and from Proposition 2.2.2 we derive some $\delta \in H$ with $\overline{\nu(g)}(\delta) = \gamma$.
As Exp_W is an open map, the intersection $S_x \cap \operatorname{Im} \operatorname{Exp}_W$ is a non-empty open set. It contains at least one non-singular point y by Newmans theorem B.2.1. Both maps coincide on the image of Exp_W, whence

$$\nu(g)(\delta.y) = \gamma.\nu(g)(y) = \nu(h)(y) = \nu(g)(y)$$

implies $\delta.y = y$. Since y is non singular, $\delta = \mathrm{id}_V$ follows. The mapping $\overline{\nu(g)}$ is a group homomorphism from which we deduce $\gamma = \mathrm{id}_{V'}$. In conclusion $\nu(g)|_{S_x} = \nu(h)|_{S_x}$ holds, whence their germs agree, proving property (R4b). The above shows that there is locally only one choice for $\nu(g)$. From this observation, one deduces that properties (R4c)-(R4d) are also valid for $\widehat{\exp_{\mathrm{Orb}}}$.
We have thus constructed a charted map $\widehat{\exp_{\mathrm{Orb}}} = (\exp_{\mathrm{Orb}}, \{ \operatorname{Exp}_W \}_{\mathcal{V}}, [P, \nu]) \in \operatorname{Orb}(\mathcal{V}, \mathcal{W})$ for some $\mathcal{W} \subseteq \mathcal{U}$. To finish the proof we need to check that every other choice of lifts yields a charted orbifold map which is equivalent to $\widehat{\exp_{\mathrm{Orb}}}$.
Let $\widetilde{\exp_{\mathrm{Orb}}} = (\exp_{\mathrm{Orb}}, \{ E_{W'} | (W', G', \psi') \in \mathcal{V}' \}, [P', \nu'])$ be another charted orbifold map whose lifts are constructed as above. Arguing as before, for each lift Exp_W, we may cover $\operatorname{Im} \operatorname{Exp}_W$ with the domains of embeddings $\mu_W^i,\ i \in I$ of orbifold charts, such that:

(a) $\operatorname{dom} \mu_W^i \neq \operatorname{dom} \mu_W^j$ for each $i \neq j$,
(b) For each i, there is a Lift $E_{W_i'}$ of $\exp'_{\mathrm{Orb}}$ and an embedding of orbifold charts λ_W^i, such that $\operatorname{Exp}_W(\operatorname{dom} \lambda_W^i) \subseteq \operatorname{dom} \mu_W^i$ and $\mu_W^i \operatorname{Exp}_{W|\operatorname{dom} \lambda_W^i} = E_{W_i'} \lambda_W^i$ hold.

Repeating this argument for each chart in $\mathcal{V}$, we obtain an orbifold atlas $\mathcal{A}$ of charts for Ω and a family $\mathcal{F}$ of orbifold charts for Q. In particular for each chart $A \in \mathcal{A}$, there is a chart in $\mathcal{F}$ together with two pairs of embeddings of orbifold charts: The first pair (ι_A^1, ι_A^2) being the canonical inclusion into $\operatorname{dom} \operatorname{Exp}_W$, respectively $\operatorname{cod} \operatorname{Exp}_W$ for a suitable lift of $\exp_{\mathrm{Orb}}$, while the second pair is fiven by the embeddings (λ_A, μ_A) constructed above. It is now easy to check, that the data $(\mathcal{A}, \mathcal{F}, (\iota_A^1, \iota_A^2)_{A \in \mathcal{A}})$ and $\mathcal{A}, \mathcal{F}, (\lambda_A, \mu_A)_{A \in \mathcal{A}}$ satisfy the hypothesis of Lemma E.4.2. By construction the induced lifts of $\widehat{\exp_{\mathrm{Orb}}}$ and $\widetilde{\exp_{\mathrm{Orb}}}$ coincide. In particular the induced lifts satisfy an identity as in (5.2.4), i.e. by construction they are given as the composition of geodesic flows change of charts morgphisms and bundle projection of a manifold. Arguing as above shows that locally there is just one choice for the change of charts in the image of ν. Local uniqueness forces $\exp_{\mathrm{Orb}} \sim \widehat{\exp_{\mathrm{Orb}}}$ by Definition E.2.5. $\square$

The above proof reveals several useful properties of the lifts for $\exp_{\mathrm{Orb}}$, which we collect in the following

5.2.4 Remark

(a) The proof of Proposition 5.2.3 shows, that arbitrary sets of lifts (which are given as lifts of orbifold geodesic flows evaluated at 1) for $\exp_{\mathrm{Orb}}$, where no two are defined on the same chart, may be complemented to a family of local lifts which satisfy (R2) of Definition E.2.3. Each of these families then induces a representative of $[\exp_{\mathrm{Orb}}]$.
(b) The families of lifts we constructed in Proposition 5.2.3 have the additional property, that for each $\operatorname{Exp}_W \colon (W, G_W, \pi) \to (U_W, G_{U_W}, \psi)$ there is an orbifold chart (V, H, φ), such that $W \subseteq TV$ is a H-stable subset which is G_W-invariant.

6. Lie group structure on the orbifold diffeomorphism group

Throughout this section, we assume that $(Q,\mathcal{U},\rho)$ is a smooth Riemannian orbifold, where the Riemannian orbfold metric is defined on the maximal atlas. We construct a Lie group structure on $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ by an application of Theorem C.4.3. To this end we first construct a Lie group structure on a subgroup.

6.1. Lie group structure on $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$

It turns out that our approach needs a framework, i.e. an orbifold atlas together with a collection of local data, which we fix now. Based on this preliminary work, we construct a Lie group structure on some subgroup $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0 \subseteq \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$. In section 6.2 this Lie group becomes the identity component for the Lie group $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$.

6.1.1 Construction

I. Choose for each connected component $C \subseteq Q$ some $z_C \in C$. As Q is locally path connected, each component of Q is open. Hence $\{ z_C | C \subseteq Q \text{ connected component} \}$ is a discrete subset. Combining Proposition 2.6.7 with Lemma 2.6.6, we may choose orbifold atlases $\mathcal{A},\mathcal{B} \subseteq \mathcal{U}$ with the following properties:
   (a) the atlases $\mathcal{A} := \{ (U_i, G_i, \psi_i) | i \in I \}, \mathcal{B} := \{ (W_j, H_j, \varphi_j) | j \in J \}$ are locally finite,
   (b) each chart in $\mathcal{A},\mathcal{B}$ is relatively compact (i.e. its image in Q is relatively compact),
   (c) For each connected component $C \subseteq Q$, there are unique $i_C \in I, j_C \in J$ with $z_C \in \psi_i(U_i)$ (resp. $z_C \in \varphi_j(W_j)$) if and only if $i = i_C$ (resp. $j = j_C$) hold,
   (d) $\mathcal{A}$ is a refinement of $\mathcal{B}$ and there is a map $\alpha\colon I \to J$ such that each $i \in I$ satisfies:
      i) $\overline{U_i} \subseteq W_{\alpha(i)}$ and the canonical inclusion of sets is an embedding of orbifold charts, implying $G_i \subseteq H_{\alpha(i)}$ and $\psi_i = \varphi_{\alpha(i)}|_{U_i}$,
      ii) $\alpha(i_C) = j_C$,
      iii) $\alpha^{-1}(j)$ is finite for each $j \in J$

II. For each $i \in I$ the set $\overline{U_i} \subseteq W_{\alpha(i)}$ is compact and connected. By local compactness, there is a relatively compact connected open set $\overline{U}_i \subseteq O_i \subseteq W_{\alpha(i)}$. The set $H_{\alpha(i)}.O_i$ is open, $H_{\alpha(i)}$-invariant and $\overline{U_i}$ is a connected subset of $O_i \subseteq H_{\alpha(i)}.O_i$. Thus $\overline{U_i}$ is contained in a connected component of $H_{\alpha(i)}.O_i$. Replacing O_i with this component, without loss of generality, O_i is an open $H_{\alpha(i)}$-stable subset. Notice that $G_i \subseteq H_{\alpha(i),O_i}$ holds by construction.

III. For each $j \in J$ define a compact H_j-invariant subset $\mathcal{K}_j := H_j.\bigcup_{i\in\alpha^{-1}(j)} \overline{O_i}$. Apply Lemma 2.6.8 with respect to the compact family $\{ \mathcal{K}_j | j \in J \}$ and the atlas $\mathcal{B}$. There is a covering for each $\mathcal{K}_j$ with a finite family $\mathcal{Z}_j := \left\{ Z_j^k \middle| 1 \leq k \leq N_j \right\}$ of open $H_{\alpha(i)}$-stable sets, such that: for each member of $\mathcal{Z}_j$ there is a finite family of embeddings $\left\{ \lambda_{jh}^k \colon Z_j^k \to W_h \middle| h \in Z(j,k) \right\}$ with

properties as in Lemma 2.6.8. By Part (c) of Lemma 2.6.8, each Z_j^k is relatively compact and the embedding λ_{jh}^k is the restriction of an embedding $\hat{\lambda}_{jh}^k$ whose domain contains $\overline{Z_j^k}$.

IV. Consider the open submanifold $\mathcal{K}_j^\circ$, which is σ-compact as an open subset of the second countable locally compact manifold W_j (cf. the proof of Proposition 2.6.7 (d)). By Lemma D.0.5, we may cover each $\mathcal{K}_j^\circ, j \in J$ with a countable family $\left\{ (V_{5,k}^j, \kappa_k^j) \middle| k \in \mathbb{N} \right\}$ of manifold charts, such that the covering is locally finite and subordinate to the open covering $\left\{ Z_k^j \cap \mathcal{K}_j^\circ \middle| 1 \leq k \leq N_j \right\}$ of $\mathcal{K}_j^\circ$. Furthermore these charts satisfy $\kappa_k^j(V_{5,j}^k) = B_5(0)$ and the families $V_{r,j}^k := (\kappa_k^j)^{-1}(B_r(0))$ cover $\mathcal{K}_j^\circ$ for $r \in [1,5]$.
Since $H_{\alpha(i)}$ is finite, the set $H_{\alpha(i)}.\overline{U}_i \subseteq \mathcal{K}_{\alpha(i)}^\circ$ is compact. The atlas $\left\{ (V_{5,k}^j, \kappa_k^j) \middle| k \in \mathbb{N} \right\}$ is locally finite, whereas there is a finite subset $\mathcal{F}_5(H_{\alpha(i)}.\overline{U_i})$ such that $V_{5,\alpha(i)}^k \cap H_{\alpha(i)}.\overline{U}_i \neq \emptyset$ if and only if the chart belongs to $\mathcal{F}_5(H_{\alpha(i)}.\overline{U_i})$. We define open sets

$$\Omega_{r,i} := \bigcup_{(V_{5,\alpha(i)},\kappa_n^{\alpha(i)}) \in \mathcal{F}_5(H_{\alpha(i)}.\overline{U_i})} V_{r,\alpha(i)}^n, \quad r \in [1,5]$$

and compact sets $K_{5,i} := \overline{\Omega_{5,i}}$. There is a finite subset $\mathcal{F}_5(K_{5,i})$ such that a chart belongs to $\mathcal{F}_5(K_{5,i})$ if and only if $V_{5,\alpha(i)}^k \cap H_{\alpha(i)}.K_{5,i} \neq \emptyset$ holds. Observe that $H_{\alpha(i)}.\overline{U_i} \subseteq \Omega_{1,i}$ is satisfied.

V. Let ρ_j be the Riemannian metric on W_j and $\exp_{W_j} \colon D_j \to W_j$ the associated Riemannian exponential map. By compactness of $\mathcal{K}_j$ and Lemma D.0.2 there are constants $s_j > 0$ for $j \in J$, such that: The closure of $\hat{O}_j := \bigcup_{x \in \mathcal{K}_j^\circ} B_{\rho_j}(0_x, s_j) \subseteq TW_j$, is contained in D_j and $\exp_{W_j}$ restricts to a diffeomorphism on $T_xW_j \cap \hat{O}_j$ for each $x \in \mathcal{K}_j$. Since $\hat{\lambda}_{jh}^k(\overline{Z_j^k})$ is compact, Lemma D.0.2 yields a constant $0 < S_{jk} < \min\{ s_h | h \in Z(j,k) \}$ such that $\exp_{W_h}$ restricts to a diffeomorphism on

$$T_{\hat{\lambda}_{jh}^k(x)} W_h \cap T\hat{\lambda}_{jh}^k \left(B_{\rho_j}(0_x, S_{jk})\right), \quad x \in \overline{Z_j^k}$$

Furthermore since change of charts are Riemannian embeddings, by choice of S_{jk}

$$T\lambda_{jh}^k \left(B_{\rho_j}(0_x, S_{jk})\right) \subseteq B_{\rho_h}(0_{\lambda_{jh}^k(x)}, s_h)$$

holds for $x \in \operatorname{dom} \lambda_{jh}^k$. For each $j \in J$ we define $S_j := \min\{ S_{jk} | 1 \leq k \leq N_j \}$.
The family $\mathcal{F}_5(K_{5,i})$ is finite and by Lemma D.0.2 for each chart $(V_{5,\alpha(i)}^k, \kappa_k^{\alpha(i)}) \in \mathcal{F}_5(K_{5,i})$ the set $\bigcup_{x \in V_{3,\alpha(i)}^k} B_{\rho_{\alpha(i)}}(0_x, S_{\alpha(i)})$ is a neighborhood of the zero-section on the compact set $\overline{V_{2,\alpha(i)}^k}$.
Hence Wallace Lemma [20, 3.2.10] yields a constant $R_i > 0$ with

$$B_2(0) \times B_{R_i}(0) \subseteq T\kappa_k^{\alpha(i)} \left(\bigcup_{x \in V_{2,\alpha(i)}^k} B_{\rho_{\alpha(i)}}(0_x, S_{\alpha(i)}) \right) \quad \forall (V_{5,\alpha(i)}^k, \kappa_k^{\alpha(i)}) \in \mathcal{F}_5(K_{5,i})$$

For the rest of this section we fix the data constructed in 6.1.1 and use the symbols without further explanation. The next Lemma is a rather technical statement. It is the first step in constructing orbifold diffeomorphisms using the Riemannian orbifold exponential map.

6.1.2 Lemma *Consider* $(U_i, G_i, \psi_i) \in \mathcal{A}$ *and for an orbisection* $[\hat{\sigma}] \in \mathfrak{X}_{Orb}(Q)$ *denote by* $\sigma_{\alpha(i)}$ *its canonical lift on* $W_{\alpha(i)}$*. There exists an open neighborhood* $\mathcal{N}_i \subseteq \mathfrak{X}\left(W_{\alpha(i)}\right)$ *of* $0_{\alpha(i)}$*, such that* $\sigma_{\alpha(i)} \in \mathcal{N}_i$ *implies the following:*

i. $\overline{\psi_i(U_i)} \subseteq \hat{\sigma}^{-1}(\Omega)$*, where* Ω *is the domain of* $\exp_{\mathrm{Orb}}$

ii. $E^\sigma|_{\psi_i(U_i)} := \exp_{\mathrm{Orb}} \circ\sigma|_{\psi_i(U_i)}^{\Omega}$ *induces a diffeomorphism of orbifolds onto its image.*

iii. $\sigma_{\alpha(i)}(\overline{\Omega_{2,i}}) \subseteq \hat{O}_{\alpha(i)}$ *holds for* $\hat{O}_{\alpha(i)}$ *as in Construction 6.1.1 V.,*

iv. there is a zero-neighborhood $\mathcal{N}_i^{\Omega_{5,i}} \subseteq \mathfrak{X}(\Omega_{5,i})$ *with* $\mathcal{N}_i = (\mathrm{res}_{\Omega_{5,i}}^{W_{\alpha(i)}})^{-1}(\mathcal{N}_i^{\Omega_{5,i}})$*.*

Proof. The set $O_i \subseteq \mathcal{K}_{\alpha(i)}^\circ$ is open $H_{\alpha(i)}$-stable, whence a $H_{\alpha(i)}$-stable open subset is given by the set $TO_i \cap \hat{O}_{\alpha(i)} \subseteq D_{\alpha(i)}$. We obtain an orbifold chart $(TO_i \cap \hat{O}_{\alpha(i)}, H_{\alpha(i),TO_i\cap\hat{O}_{\alpha(i)}}, T\psi_{\alpha(i)}|_{TO_i\cap\hat{O}_{\alpha(i)}})$ together with a lift of $\exp_{\mathrm{Orb}}$: $\mathrm{Exp}_{TO_i\cap\hat{O}_{\alpha(i)}} := \exp_{W_{\alpha(i)}}|_{TO_i\cap\hat{O}_{\alpha(i)}} \colon TO_i \cap \hat{O}_{\alpha(i)} \to W_{\alpha(i)}$. By Remark 5.2.4 (a), there is a representative $\widehat{\exp_{\mathrm{Orb}}} \in \mathrm{Orb}(\mathcal{V},\mathcal{U})$ of $\exp_{\mathrm{Orb}}$, such that Exp_{Ω_i} is contained in the family of local lifts of $\widehat{\exp_{\mathrm{Orb}}}$. Notice that $\psi_i(U_i) \subseteq Q$ is an open subset, whose inclusion $\iota_{\psi_i(U_i)}$ induces an open suborbifold structure (see Definition 3.2.1). Consider an orbisection $[\hat{\sigma}]$ with $\mathrm{Im}\,\sigma|_{\psi_i(U_i)} \subseteq \Omega$. Remark 3.2.4 (b) implies that there is a well-defined map of orbifolds $\left[\hat{E}^\sigma|_{\psi_i(U_i)}\right] := [\exp_{\mathrm{Orb}}] \circ [\hat{\sigma}]|_{\psi_i(U_i)}^{\Omega}$. Following these preliminary remarks, we proceed in several steps:

Step 1: Apply Lemma D.0.7 to the family $\mathcal{F}_5(H_{\alpha(i)}.\overline{U_i})$ to obtain an open zero-neighborhood $N_i^{\Omega_{5,i}} \subseteq \mathfrak{X}(\Omega_{5,i})$. By construction $0_{\alpha(i)} \in N_i := (\mathrm{res}_{\Omega_{5,i}}^{W_{\alpha(i)}})^{-1}(N_i^{\Omega_{5,i}}) \subseteq \mathfrak{X}\left(W_{\alpha(i)}\right)$ and the following conditions hold: For each $X \in N_i$ the map $\exp_{W_{\alpha(i)}} \circ X|_{\Omega_{2,i}}$ is an open embedding into $W_{\alpha(i)}$. The set $\overline{\Omega_{2,i}} \subseteq \Omega_{5,i} \subseteq \mathcal{K}_{\alpha(i)}^\circ$ is compact, which allows the construction of a C^0-neighborhood of the zero section $P_{1,i} \subseteq \mathfrak{X}(\Omega_{5,i})$, such that $X \in P_{1,i}$ implies $X(\overline{\Omega_{2,i}}) \subseteq \hat{O}_{\alpha(i)}$. Set $\mathcal{N}_i^{\Omega_{5,i}} := N_i^{\Omega_{5,i}} \cap P_{1,i}$ and $\mathcal{N}_i := (\mathrm{res}_{\Omega_{5,i}}^{W_{\alpha(i)}})^{-1}(N_i^{\Omega_{5,i}} \cap P_{1,i})$. Each vector field in $\mathcal{N}_i$ satisfies iii. and $\mathcal{N}_i$ is a neighborhood as required in iv.. By construction $\overline{\psi_i(U_i)} = \varphi_{\alpha(i)}(\overline{U_i}) \subseteq \varphi_{\alpha(i)}(O_i)$ holds and $\mathrm{Exp}_{TO_i\cap\hat{O}_{\alpha(i)}}$ is a lift of $\exp_{\mathrm{Orb}}$, whence i. follows from property iii..
In addition for each $\sigma_{\alpha(i)} \in \mathcal{N}_i$ the map $\exp_{W_{\alpha(i)}} \circ\sigma_{\alpha(i)}|_{H_{\alpha(i)}.U_i}$ is an open embedding. Specializing to U_i, the map $e^{\sigma_i} := \mathrm{Exp}_{TO_i\cap\hat{O}_{\alpha(i)}} \circ \sigma_{\alpha(i)|U_i} = \mathrm{Exp}_{TO_i\cap\hat{O}_{\alpha(i)}} \circ \sigma_i$ is a smooth embedding. From now on consider $[\hat{\sigma}] \in \mathfrak{X}_{\mathrm{Orb}}(Q)$ with $\sigma_{\alpha(i)} \in \mathcal{N}_i$ holds.

Step 2: *The map* e^{σ_i} *is equivariant with respect to the inclusion* $\nu \colon G_i \hookrightarrow H_{\alpha(i)}$: Consider a $H_{\alpha(i)}$-invariant subset $R \subseteq \Omega_{2,i}$. We claim that $\exp_{W_{\alpha(i)}} \sigma_{W_{\alpha(i)}}|_R$ is equivariant with respect to $H_{\alpha(i)}$. If this were correct then e^{σ_i} commutes with any $\delta \in H_{\alpha(i),U_i} = G_i$, as $H_{\alpha(i)}.U_i \subseteq \Omega_{2,i}$ is invariant. To prove the claim let $\delta \in H_{\alpha(i)}$ be arbitrary and $x \in R$. As $\delta.x \in R \subseteq \Omega_{2,i}$ holds, $\sigma_{\alpha(i)}$ is a canonical lift and $H_{\alpha(i)}$ acts by Riemannian isometries, we compute: $\exp_{W_{\alpha(i)}} \sigma_{\alpha(i)}(\delta.x) = \exp_{W_{\alpha(i)}} T\delta\sigma_{\alpha(i)}(x) = \delta.\exp_{W_{\alpha(i)}} \sigma_{\alpha(i)}(x)$, thus proving the claim. The map e^{σ_i} is a local lift of $\exp_{\mathrm{Orb}} \circ\sigma|_{\psi_i(U_i)}$. Since composition in **Orb** is well-defined, a representative of $[\exp_{\mathrm{Orb}}] \circ [\hat{\sigma}]|_{\psi_i(U_i)}^{\Omega}$ is given by $E^\sigma|_{\psi_i(U_i)} = (\exp_{\mathrm{Orb}} \circ\sigma|_{\psi_i(U_i)}, e^{\sigma_i}, G_i, \nu) \in \mathrm{Orb}(\{(U_i, G_i, \psi_i)\}, \{(W_{\alpha(i)}, H_{\alpha(i)}, \varphi_{\alpha(i)})\})$.

Step 3: *The set* $\operatorname{Im} e^{\sigma_i}$ *is* $H_{\alpha(i)}$*-stable with* $H_{\alpha(i)}.\operatorname{Im} e^{\sigma_i} \subseteq \Omega_{2,i}$: Consider $\delta \in H_{\alpha(i)}$, such that $\delta.\operatorname{Im} e^{\sigma_i} \cap \operatorname{Im} e^{\sigma_i} \neq \emptyset$ holds. For $x, y \in U_i$ with $e^{\sigma_i}(x) = \delta.e^{\sigma_i}(y)$ one obtains

$$\exp_{W_{\alpha(i)}} \circ \sigma_{\alpha(i)}(x) = e^{\sigma_i}(x) = \delta.e^{\sigma_i}(y) = \exp_{W_{\alpha(i)}} \sigma_{\alpha(i)}(\delta.y).$$

From Step 1 we conclude $x = \delta.y$, since on $H_{\alpha(i)}.U_i \subseteq \Omega_{2,i}$ the map $\exp_{W_{\alpha(i)}} \circ \sigma_{\alpha(i)}$ is a smooth embedding. By $H_{\alpha(i)}$-stability of U_i, $\delta \in G_i$ holds, whence $\delta.\operatorname{Im} e^{\sigma_i} = \operatorname{Im} e^{\sigma_i}$ follows. This proves the $H_{\alpha(i)}$-stability of $\operatorname{Im} e^{\sigma_i}$ and $G_{\operatorname{Im} e^{\sigma_i}} = G_i$.
The canonical lift $\sigma_{\alpha(i)}$ is contained in $\mathcal{N}_i$. By construction of $\Omega_{1,i}$ (cf. Lemma D.0.7), the equivariance of map implies:

$$H_{\alpha(i)}.\operatorname{Im} e^{\sigma_i} = \exp_{W_{\alpha(i)}} \sigma_i(H_{\alpha(i)}.U_i) \subseteq \exp_{W_{\alpha(i)}} \sigma_i(\Omega_{1,i}) \subseteq \Omega_{2,i}.$$

Step 4: $E^\sigma|_{\psi_i(U_i)}$ *is injective and a homeomorphism onto its open image*:
Consider $x, y \in \psi_i(U_i)$ with $E^\sigma|_{\psi_i(U_i)}(x) = E^\sigma|_{\psi_i(U_i)}(y)$ and choose preimages $z_x \in \psi_i^{-1}(x), z_y \in \psi_i^{-1}(y)$ of x respectively y in U_i. Since e^{σ_i} is a lift of $E^\sigma|_{\psi_i(U_i)}$, there exists $\delta \in H_{\alpha(i)}$ such that $e^{\sigma_i}(z_x) = \delta.e^{\sigma_i}(z_y)$. By Step 3 we must have $\delta \in G_i$. Since e^{σ_i} is an embedding, equivariance of this map yields $\delta.z_y = z_x$. Both points are in the same orbit, which forces x and y to coincide. Hence $E^\sigma_{|\tilde{U}_i}$ is injective.
The local lift e^{σ_i} is a smooth embedding with open image and the maps of orbifold charts are continuous and open. For any open subset $S \subseteq \psi_i(U_i)$, $E^\sigma|_{\psi_i(U_i)} = \varphi_{\alpha(i)} \circ e^{\sigma_i} \circ \psi_i^{-1}(S)$ is an open set. In conclusion $E^\sigma|_{\psi_i(U_i)}$ is an open map, whose image is open in Q. In particular $\operatorname{Im} E^\sigma|_{\psi_i(U_i)}$ is an open suborbifold of Q. An atlas for $\operatorname{Im} E^\sigma|_{\psi_i(U_i)}$ is given by $\left\{ (\operatorname{Im} e^{\sigma_i}, G_i, \varphi_{\alpha(i)}|_{\operatorname{Im} e^{\sigma_i}}) \right\}$.
The map $E^\sigma|_{\psi_i(U_i)}$ is a homeomorphism mapping the open suborbifold $\psi_i(U_i)$ of Q onto an open suborbifold, such that the local lift of $E^\sigma|_{\psi_i(U_i)}$ is a diffeomorphism onto its (open) image. Proposition 3.1.10 assures that $[\hat{E}^\sigma|_{\psi_i(U_i)}]$ is a diffeomorphism of orbifolds. □

6.1.3 Later on, we shall apply patched mapping techniques (cf. Section C.3) to prove the smoothness of several maps. To do so, we have to define an orbifold atlas, where charts may occur severalfold: Let $\mathcal{C} := \left\{ (W_{\alpha(i)}, H_{\alpha(i)}, \varphi_{\alpha(i)}) \middle| i \in I \right\}$ be the orbifold atlas which arises from $\mathcal{B}$ by collecting a copy of $(W_j, H_j, \varphi_j) \in \mathcal{B}$ for each $i \in \alpha^{-1}(j)$. Observe that this atlas is locally finite and each chart is relatively compact, as $\alpha^{-1}(j)$ is finite and $\mathcal{B}$ is locally finite with relatively compact charts.

6.1.4 Proposition *There are open zero-neighborhoods* $\mathcal{N}_i \subseteq \mathfrak{X}\left(W_{\alpha(i)}\right)$, $i \in I$ *which generate an open zero-neighborhood* $\mathcal{N} \subseteq \mathfrak{X}_{Orb}\left(Q\right)_c$, *such that each* $[\hat{\sigma}] \in \mathcal{N}$ *induces an orbifold diffeomorphism* $[\hat{E}^\sigma] := [\exp_{\mathrm{Orb}}] \circ [\hat{\sigma}]|^\Omega \in \operatorname{Diff}_{\mathrm{Orb}}\left(Q, \mathcal{U}\right)$.

Proof. For each $i \in I$ construct via Lemma 6.1.2 a neighborhood $\mathcal{N}_i \subseteq \mathfrak{X}\left(W_{\alpha(i)}\right)$. The construction shows that for each $[\hat{\sigma}]$ with $\sigma_{\alpha(i)} \in \mathcal{N}_i$, the map $E^\sigma|_{\psi_i(U_i)}$ is an embedding of the open suborbifold $\psi_i(U_i)$. By definition of the direct sum topology, the box $\bigoplus_{i\in I} \mathcal{N}_i := \left(\prod_{i \in I} \mathcal{N}_i\right) \bigcap \bigoplus_{i \in I} \mathfrak{X}\left(W_{\alpha(i)}\right)$ is an open subset of $\bigoplus_{i \in I} \mathfrak{X}\left(W_{\alpha(i)}\right)$ (cf. [37, 4.3] respectively [24, Proposition 7.1] for a proof).

Using the atlas $\mathcal{C}$ introduced above, we define the set

$$\mathcal{N} := \Lambda_{\mathcal{C}}^{-1}\left(\bigoplus_{i\in I}\mathcal{N}_i\right), \tag{6.1.1}$$

which is open in the c.s. orbisection topology by Lemma 4.3.4. A combination of Definition 3.2.3 and Remark 5.2.4 (a) shows that $[\hat{\sigma}]$ in $\mathcal{N}$ induces a well-defined map of orbifolds $[\hat{E}^\sigma] := [\exp_{\mathrm{Orb}}]\circ[\hat{\sigma}]|^{\Omega}$, such that $E^\sigma := \exp_{\mathrm{Orb}}\circ\sigma\colon Q\to Q$ is a local homeomorphism. In particular $E^\sigma|_{\psi_i(U_i)}$ is an open embedding for each $i\in I$. Let $\widehat{\exp_{\mathrm{Orb}}}$ be the representative of the Riemannian orbifold exponential map obtained from the family $\left\{\mathrm{Exp}_{TO_i\cap\hat{O}_{\alpha(i)}}\right\}_{i\in I}$ by Remark 5.2.4 (a) and $\hat{\sigma}|^{\Omega}\in\mathrm{Orb}(\mathcal{A},\mathcal{TA}|_{\Omega})$ the representative of $[\hat{\sigma}]|^{\Omega}$ obtained by corestriction of each σ_i to the set $TO_i\cap\hat{O}_{\alpha(i)}$ for $i\in I$. As composition in **Orb** is well-defined, we obtain $[\widehat{\exp_{\mathrm{Orb}}}\circ\hat{\sigma}|^{\Omega}] = [\exp_{\mathrm{Orb}}]\circ[\hat{\sigma}]|^{\Omega}$. Hence the lifts constructed in Lemma 6.1.2 yield a representative $\hat{E}^\sigma := \widehat{\exp_{\mathrm{Orb}}}\circ\hat{\sigma} = (E^\sigma,\{e^{\sigma_i}|i\in I\},P,\nu)\in\mathrm{Orb}(\mathcal{A},\mathcal{C})$. Here each lift e^{σ_i} is a smooth embedding and (P,ν) is obtained by an application of Construction E.4.1. The image of such a lift is an orbifold chart $(\mathrm{Im}\, e^{\sigma_i}, G_i, \varphi_{\alpha(i)}|_{\mathrm{Im}\, e^{\sigma_i}})$.
We have to check that E^σ is surjective and injective for every $[\hat{\sigma}]\in\mathcal{N}$ to prove the assertion. Reviewing the construction of $\mathcal{N}_i$, the map E^σ maps $\psi_i(U_i)$ to $\varphi_{\alpha(i)}(W_{\alpha(i)})$. Every orbifold chart is a connected set, whence its image is contained in a connected component of Q. Thus E^σ maps every connected component of Q into itself. Bijectivity of the map may therefore be checked for each component separately and we shall assume Q to be connected.
As a first step, we claim that for every orbisection $[\hat{\sigma}]\in\mathcal{N}$ the map E^σ is a proper map. To this end consider an arbitrary compact subset $L\subseteq Q$. The atlas $\mathcal{B}$ is locally finite and thus L meets only finitely many of the sets $\varphi_j(W_j), j\in J$, say $L\subseteq\bigcup_{r=1}^n\varphi_{j_r}(W_{j_r})$ and $L\cap\varphi_j(W_j)=\emptyset$ for all $j\in J\setminus\{j_1,\ldots,j_n\}$. For $[\hat{\sigma}]\in\mathcal{N}$ we have $E^\sigma(\overline{\psi_i(U_i)})\subseteq\varphi_{\alpha(i)}(W_{\alpha(i)})$. The closed set $(E^\sigma)^{-1}(L)$ is thus contained in

$$(E^\sigma)^{-1}(L)\subseteq\bigcup_{r=1}^n\bigcup_{i\in\alpha^{-1}(j_r)}\overline{\varphi_i(U_i)}. \tag{6.1.2}$$

By construction 6.1.1 each $\alpha^{-1}(j_r)$ is a finite set. Hence $(E^\sigma)^{-1}(L)$ is compact as a closed subset of a union of finitely many compact sets. Since L was arbitrary, E^σ is a proper map (cf. [8, §10 3. Proposition 7]).
Combining the fact that Q is locally compact by Proposition 2.4.3 and E^σ being a proper map, E^σ is a closed map (cf. [8, §10 1. Theorem 1]). The image of E^σ is an open and closed set, since images of local homeomorphisms are open. But Q is connected and thus E^σ has to be surjective.
The map E^σ is a proper, surjective local homeomorphism of connected and path-connected locally compact spaces. Summing up, E^σ is a covering of Q onto Q by [21, Theorem 4.22]. Recall 6.1.1 I. (c): There is some $z_Q\in Q$ such that z_Q is contained in a unique pair of orbifold charts $\big((U_{z_Q},G_{z_Q},\psi_{z_Q}),(W_{z_Q},H_{z_Q},\varphi_{z_Q})\big)\in\mathcal{A}\times\mathcal{B}$. Since $E^\sigma(\psi_i(U_i))\subseteq\varphi_{\alpha(i)}(W_{\alpha(i)})$ and z_Q is not contained in any $\varphi_{\alpha(i)}(W_j)$ except for $j=z_Q$ by 6.1.1 we derive from (6.1.2): $|(E^\sigma)^{-1}(z_0)|=1$. The number of sheets of a covering is an invariant for the connected space Q (cf. [21, Theorem 4.16]), whence E^σ is injective.
In conclusion we have constructed a charted orbifold map $\hat{E^\sigma}$, such that E^σ is a continuous, closed bijective map (i.e. a homeomorphism by [19, III. Theorem 12.2]) and each lift $e^{\sigma_{V_i}}, (V_i,G_i,\psi_i)\in\mathcal{V}$ is a smooth embedding with an open image. Each lift is a local diffeomorphism, whence Proposition 3.1.10 implies that $\hat{E^\sigma}$ is a representative of an orbifold diffeomorphism $[\hat{E^\sigma}] = [\exp_{\mathrm{Orb}}]\circ[\hat{\sigma}]|^{\Omega}$. $\square$

6.1.5 Proposition *Consider the family* $\{ \mathcal{N}_i \}_{i \in I}$ *as in Proposition 6.1.4. For each* $i \in I$ *there is an open neighborhood* $P_{2,i} \subseteq \mathfrak{X}(\Omega_{5,i})$ *of the zero section and sets* $\mathcal{M}_i^{\Omega_{5,i}} := \mathcal{N}_i^{\Omega_{5,i}} \cap P_{2,i}$, $\mathcal{M}_i := (\mathrm{res}_{\Omega_{5,i}}^{W_{\alpha(i)}})^{-1}(\mathcal{M}_i^{\Omega_{5,i}})$ *such that on the zero-neighborhood* $\mathcal{M} := \Lambda_{\mathcal{C}}^{-1}\left(\bigoplus_{i\in I}\mathcal{M}_i\right)$ *the map*

$$E\colon \mathcal{M} \to \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U}),\ E([\hat{\sigma}]) := [\hat{E}^\sigma] = [\exp_{\mathrm{Orb}}] \circ [\hat{\sigma}]|^{\Omega},$$

is injective with $E(\mathbf{0}_{Orb}) = \mathrm{id}_{(Q,\mathcal{U})}$.

Proof. Following Proposition 6.1.4 each $[\hat{\sigma}] \in \mathcal{N} = \Lambda_{\mathcal{C}}^{-1}\left(\bigoplus_{i\in I}\mathcal{N}_i\right)$ induces an orbifold diffeomorphism $[\hat{E}^\sigma]$. Shrink $\mathcal{N}_i$ to obtain an open C^1-neighborhood $\mathcal{M}_i$ of the zero-section in $\mathfrak{X}\left(W_{\alpha(i)}\right)$: Choose for each $i \in I$ a non-singular point $z_i \in U_i$ (which exists due to Newmans Theorem B.2.1, since U_i is an open set) and a $H_{\alpha(i)}$-stable z_i-neigborhood $U_{z_i} \subseteq W_{\alpha(i)}$ with $H_{\alpha(i),U_{z_i}} = \left\{ \mathrm{id}_{W_{\alpha(i)}} \right\}$. This is possible since z_i is non singular. The family $\mathcal{F}_5(H_{\alpha(i)}.\overline{U_i})$ constructed in 6.1.1 covers $\overline{U_i}$ and we may choose a chart $(V_{5,\alpha(i)}^k, \kappa_k^{\alpha(i)})$, such that $z_i \in V_{3,\alpha(i)}^k$. Consider the open set $\hat{U}_{z_i} := TV_{5,\alpha(i)}^k \cap \hat{O}_{\alpha(i)} \cap \exp_{W_{\alpha(i)}}^{-1}(U_{z_i}) \subseteq TW_{\alpha(i)}$. The intersection $T_{z_i}W_{\alpha(i)} \cap \hat{U}_{z_i}$ is an open zero-neighborhood. We obtain another open zero-neighborhood

$$\lfloor \kappa_k^{\alpha(i)}(z_i), \mathrm{pr}_2(T\kappa_k^{\alpha(i)}(\hat{U}_{z_i})\rfloor \subseteq C^\infty(B_5(0),\mathbb{R}^d)$$

where $\mathrm{pr}_2\colon B_5(0) \times \mathbb{R}^d \to \mathbb{R}^d$ is the projection. Define $P_{2,i} \subseteq \mathfrak{X}(\Omega_{5,i})$ to be the open zero-neighborhood induced by $\lfloor \kappa_k^{\alpha(i)}(z_i), \mathrm{pr}_2(T\kappa_k^{\alpha(i)}(\hat{U}_{z_i})\rfloor$. By construction $\exp_{W_{\alpha(i)}} \circ \sigma_{\alpha(i)}$ maps z_i into U_{z_i} if $\sigma_{\alpha(i)}$ is contained in $P_{2,i}$. The intersection $\mathcal{M}_i^{\Omega_{5,i}} := \mathcal{N}_i^{\Omega_{5,i}} \cap P_{2,i}$ is a non-empty open zero-neighborhood in $\mathfrak{X}(\Omega_{5,i})$. Define $\mathcal{M}_i := (\mathrm{res}_{\Omega_{5,i}}^{W_{\alpha(i)}})^{-1}(\mathcal{M}_i^{\Omega_{5,i}}) \subseteq \mathcal{N}_i$, then $\mathcal{M} := \Lambda_{\mathcal{C}}^{-1}\left(\bigoplus_{i\in I}\mathcal{M}_i\right)$ contains $\mathbf{0}_{\mathrm{Orb}}$ and is an open subset of $\mathcal{N}$ in $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$.
We claim that the map E (as in the statement of the Proposition) is injective on $\mathcal{M}$. Assume that there are $[\hat{\sigma}], [\hat{\tau}] \in \mathcal{M}$, such that $E([\hat{\sigma}]) = E([\hat{\tau}])$. For $E([\hat{\sigma}]) = [\hat{E}^\sigma]$ there is representative $\hat{E}^\sigma$ in $\mathrm{Orb}(\mathcal{A},\mathcal{C})$ by Proposition 6.1.4. By assumption the orbifold maps induced by $\hat{E}^\sigma$ and $\hat{E}^\tau$ coincide, whence $E^\tau = E^\sigma$ follows. We will prove that for each $i \in I$ the lifts e^{σ_i} and e^{τ_i} coincide. Fix $i \in I$ and observe that $E^\sigma = E^\tau$ implies that for each $z \in U_i$ there is some $\gamma_z \in H_{\alpha(i)}$ with $e^{\sigma_i}(z) = \gamma_z.e^{\tau_i}(z)$. Consider a component C of $U_i \setminus \Sigma_{G_i}$. The set $\{ c \in C | \gamma.e^{\sigma_i}(c) = e^{\tau_i}(c) \}$ is an open and closed subset of C. As C is connected, there is a unique $\gamma_C \in H_{\alpha(i)}$ with $e^{\sigma_i}|_{\overline{C}} = \gamma_C e^{\tau_i}|_{\overline{C}}$. For $x \in \overline{C} \cap \overline{C'}$ this yields the identity $T_x\gamma_C e^{\tau_i} = T_x e^{\sigma_i} = T_x \gamma_{C'} e^{\tau_i}$. Since e^{τ_i} is a diffeomorphism, we derive $T_{e^{\tau_i}(x)}\gamma_{C'}^{-1}\gamma_C = T_{e^{\tau_i}(x)} \mathrm{id}_{W_{\alpha(i)}}$ and $\gamma_{C'}^{-1}\gamma_C \in H_{\alpha(i),e^{\tau_i}(x)}$. By [48, Lemma 2.10] $\gamma_C = \gamma_{C'}$ follows, whence there is a unique γ with $\gamma.e^{\tau_i} = e^{\sigma_i}$. Specializing, we obtain $\gamma.e^{\tau_i}(z_i) = e^{\sigma_i}(z_i)$. The lifts $\sigma_{\alpha(i)}, \tau_{\alpha(i)}$ are elements of $\mathcal{M}_i$, whence by definition of $\mathcal{M}_i$, $e^{\sigma_i}(z_i), e^{\tau_i}(z_i) \in U_{z_i}$ holds. The $H_{\alpha(i)}$-stability of U_{z_i} forces γ to be in the isotropy subgroup of U_{z_i}. Hence $\gamma = \mathrm{id}_{W_{\alpha(i)}}$ holds and we obtain $\exp_{W_{\alpha(i)}} \circ \sigma_i = \exp_{W_{\alpha(i)}} \circ \tau_i$. Lemma 6.1.2 iii. implies that $\mathrm{Im}\,\sigma_i$ and $\mathrm{Im}\,\tau_i$ are contained in $\hat{O}_{\alpha(i)}$. As $\exp_{W_{\alpha(i)}}$ is injective on $T_xW_{\alpha(i)} \cap \hat{O}_{\alpha(i)}$ for $x \in U_i$, we must have $\tau_i = \sigma_i$. Repeating the argument for $i \in I$, the families $\{ \tau_i \}_{i\in I}$ and $\{ \sigma_i \}_{i\in I}$ coincide. As those lifts are canonical lifts, Remark 4.2.10 (a) implies $[\hat{\sigma}] = [\hat{\tau}]$ and $E\colon \mathcal{M} \to \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ is injective. □

We will apply the results of section D to construct a neighborhood $\mathcal{H}$ of the zero-orbisection:

6.1.6 Construction Using the local data obtained in Construction 6.1.1 IV., we define open sets

$$\Omega_{r,K_{5,i}} := \bigcup_{(V^n_{5,\alpha(i)},\kappa_n^{\alpha(i)})\in\mathcal{F}_5(K_{5,i})} V^n_{r,\alpha(i)}, \quad r\in[1,5].$$

By construction $\Omega_{5,i} \subseteq \overline{\Omega_{5,i}} = K_{5,i} \subseteq \Omega_{r,K_{5,i}}$ holds for each $r\in[1,5]$.
In Proposition 6.1.5 we have constructed sets $\mathcal{M}_i^{\Omega_{5,i}}$ as intersections $\mathcal{M}_i^{\Omega_{5,i}} = N_i^{\Omega_{5,i}} \cap P_{1,i} \cap P_{2,i}$, where $N_i^{\Omega_{5,i}}$ is an open zero-neighborhood as in Lemma D.0.7. Apply Construction D.0.8 with R_i (see Construction 6.1.1 V.) taking the role of R and $P := P_{1,i}\cap P_{2,i}$ to construct an open zero-neighborhood $\mathcal{H}_{R_i} \subseteq \mathcal{M}_i \subseteq \mathfrak{X}\left(W_{\alpha(i)}\right)$. By construction $\mathcal{H}_{R_i} = \left(\mathrm{res}^{W_{\alpha(i)}}_{\Omega_{5,K_{5,i}}}\right)^{-1}(\mathcal{H}^{K_{5,i}}_{R_i})$ holds for an open zero neighborhood $\mathcal{H}^{K_{5,i}}_{R_i} \subseteq \mathfrak{X}\left(\Omega_{5,K_{5,i}}\right)$. Finally for each $i\in I$ the construction yields a constant, $0<\tau_i<R_i$ with the following property:
Given $X \in \mathfrak{X}\left(W_{\alpha(i)}\right)$ such that for each $(V^k_{5,\alpha(i)},\kappa_k^{\alpha(i)}) \in \mathcal{F}_5(K_{5,i})$ the local representative X_k satisfies $\|X_k\|_{\overline{B_1(0)},1} \leq \tau_i$ then X is contained in $\mathcal{M}_i$.
Recall from Construction D.0.8 that for each pair $(X,Y)\in\mathcal{H}_{R_i}\times\mathcal{H}_{R_i}$ there are unique vector fields $X\diamond_i Y, X^{*_i}, Y^{*_i} \in \mathfrak{X}\left(\Omega_{\frac{5}{4},K_{5,i}}\right)$. Together with the definition of R_i (6.1.1 V.), the estimates (D.0.14) and (D.0.16) imply the following properties, which we note here for later use:

$$X\diamond_i Y(x), X^{*_i}(x) \in B_{\rho_{\alpha(i)}}(0_x, S_{\alpha(i)}) \subseteq \hat{O}_{\alpha(i)}, \quad \forall x\in\Omega_{\frac{5}{4},K_{5,i}}. \tag{6.1.3}$$

Define the open subset $\mathcal{H} := \Lambda_{\mathcal{C}}^{-1}\left(\bigoplus_{i\in I}\mathcal{H}_{R_i}\right)$ of $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$. By construction the inclusions $\mathbf{0}_{\mathrm{Orb}} \in \mathcal{H}\subseteq\mathcal{M}$ hold.

The vector fields $X\diamond_i Y$ and X^{*_i} induced by orbisections in $\mathcal{H}$ yield families, whose members are λ-related for suitable change of orbifold charts λ. The details are checked in the next Lemma.

6.1.7 Lemma *Consider orbisections $[\hat\sigma],[\hat\tau]\in\mathcal{H}$ with families of canonical lifts $(\sigma_j)_{j\in J},(\tau_j)_{j\in J}$ with respect to the atlas $\mathcal{B}$. Let $\lambda\in\mathcal{C}h_{W_k,W_l}$ be a change of charts which satisfies $\mathrm{dom}\,\lambda\subseteq\Omega_{\frac{5}{4},K_{5,i}}$ and $\mathrm{cod}\,\lambda\subseteq\Omega_{\frac{5}{4},K_{5,j}}$ for $k=\alpha(i)$ and $l=\alpha(j)$. The following identities hold:*

$$T\lambda(\sigma_k\diamond_i\tau_k)|_{\mathrm{dom}\,\lambda} = (\sigma_l\diamond_j\tau_l)\circ\lambda \tag{6.1.4}$$
$$T\lambda\sigma_k^{*_i}|_{\mathrm{dom}\,\lambda} = \sigma_l^{*_i}\circ\lambda \tag{6.1.5}$$

In particular the maps $\sigma_j\diamond_i\tau_j|_{U_i}$ and $\sigma_j^{*_i}|_{U_i}$ are equivariant with respect to the derived action of G_i.

Proof. The identities (6.1.4) and (6.1.5) may be checked locally. Fix $x\in\mathrm{dom}\,\lambda\subseteq\Omega_{\frac{5}{4},K_{5,i}}$ together with a chart $(V^n_{5,k},\kappa_n^k)\in\mathcal{F}_5(K_{5,i})$, such that $x\in V^n_{\frac{5}{4},k}$ holds. The manifold atlas chosen for $\mathcal{K}_k^\circ\subseteq W_k$ is subordinate to the covering $(Z_k^r\cap\mathcal{K}_k^\circ)_{1\leq r\leq N_k}$. Hence there is some Z_r^k with $V^n_{5,k}\subseteq Z_r^k$. As $x\in V^n_{5,k}\subseteq\mathcal{K}_i$ and $\lambda(x)\in\Omega_{\frac{5}{4},K_{5,i}}\subseteq\mathcal{K}_j$, by construction 6.1.1 (cf. Lemma 2.6.8), there is an open embedding of orbifold charts $\mu\colon Z_k^r\to W_l$ with $\mu(x)=\lambda(x)$. After possibly replacing μ with $\gamma.\mu$

for suitable $\gamma \in H_l$, there is an open neighborhood U_x with $\mu|_{U_x} = \lambda|_{U_x}$. By construction we obtain $\mu(x) = \lambda(x) \in \Omega_{\frac{5}{4},K_{5,j}} \subseteq K_l^\circ$ and $T_x\mu = T_x\lambda$ holds. The definition of S_k together with equation (6.1.3) implies $T\mu(\sigma_j \diamond_i \tau_j)(x), T\mu\sigma_j^{*_i}(x) \in \hat{O}_l$ and $(\sigma_l \diamond_j \tau_l)\mu(x), \sigma_l^{*_j}\mu(x) \in \hat{O}_l$. Let $\exp_n$ be the Riemannian exponential map induced by the pullback metric on $B_5(0)$ with respect to κ_n^k. On $V_{5,k}^n$, the map $\mu(\kappa_n^k)^{-1}$ yields a Riemannian embedding of $B_5(0)$ into W_l. From [41, IV. Proposition 2.6], we deduce $\exp_{W_j} T\mu(\kappa_n^k)^{-1}(v) = \mu(\kappa_n^k)^{-1}\exp_n(v)$ for each $v \in \operatorname{dom}\exp_n$. Recall from Construction 6.1.6 that for $i \in I$, there is some open set $\mathcal{H}_{R_i}$ with the same properties as in Lemma D.0.7, such that $[\hat{\sigma}] \in \mathcal{H}$ implies $\sigma_k \in \mathcal{H}_{R_i}$. For $X \in \mathcal{H}_{R_i}$ the following estimates and identities are available:

i. $\kappa_n^k \exp_{W_k} \circ X(z) = \exp_n T\kappa_n X(z)$ for each $z \in V_{3,k}^n$ (combine Lemma D.0.6 (b) and (f)),
ii. $\exp_{W_k} \circ X(\overline{V_{\frac{5}{4},k}^n}) \subseteq V_{2,k}^n$ and $\exp_{W_k} \circ X(\overline{V_{2,k}^n}) \subseteq V_{3,k}^n$, (Lemma D.0.6 (d)),
iii. $V_{\frac{5}{4},k}^n \subseteq \exp_{W_k} \circ X(V_{2,k}^n)$ (Lemma D.0.6 (d)).

The families (σ_k) and (τ_k) are canonical families, whence $\sigma_l\mu = T\mu\sigma_k$ holds. In addition on $V_{\frac{5}{4},k}^n$ the local identities (D.0.13) and (D.0.18) are available. Combining these facts we compute:

$$\begin{aligned}
\exp_{W_l} T_x\lambda(\sigma_k \diamond_i \tau_k)(x) &= \exp_{W_l} T_x\mu(\sigma_k \diamond_i \tau_k)(x) \overset{ii.}{=} \exp_{W_l} T\mu(\kappa_n^k)^{-1}\kappa_n^k(\sigma_k \diamond_i \tau_k)(x) \\
&= \mu(\kappa_n^k)^{-1}\exp_n T\kappa_n^k(\sigma_k \diamond_i \tau_k)(x) \overset{i.}{=} \mu(\kappa_n^k)^{-1}\kappa_n^k \exp_{W_k}(\sigma_k \diamond_i \tau_k)(x) \\
&\overset{(D.0.13)}{=} \mu \exp_{W_k}(\exp_{W_k}|_{N_x})^{-1}\exp_{W_k}\sigma_k \exp_{W_k}\tau_k(x) \\
&\overset{i.}{=} \exp_{W_l} T\mu\sigma_k \exp_{W_k}\tau_k(x) = \exp_{W_l}\sigma_l\mu\exp_{W_k}\tau_k(x) \\
&\overset{i.}{=} \exp_{W_l}\sigma_l \exp_{W_l}\tau_l\mu(x) = (\exp_{W_l}\sigma_l\exp_{W_l}\tau_l)\lambda(x) \\
&\overset{(D.0.17)}{=} \exp_{W_l}(\sigma_l \diamond_j \tau_l)(\lambda(x))
\end{aligned}$$

Since $\exp_{W_l}$ restricts to a diffeomorphism on $T_{\lambda(x)}W_l \cap \hat{O}_l$, the computation yields (6.1.4).
To obtain (6.1.5), we use $x \in V_{\frac{5}{4},k}^n$ and compute with the facts from above:

$$\exp_{W_l} T_x\lambda\sigma_k^{*_i}(x) = \exp_{W_l} T\mu\sigma_k^{*_i}(x) \overset{(D.0.18)}{=} \mu(\exp_{W_k} \circ\sigma_k|_{\Omega_{2,K_{5,i}}})^{-1}(x)$$

As $x \in V_{\frac{5}{4},n}^k$ holds, by iii. the image $(\exp_{W_k} \circ\sigma_k|_{\Omega_{2,K_{5,i}}})^{-1}(x)$ is contained in $V_{2,k}^n$. Therefore Construction 6.1.1 V. implies $\sigma_l\mu(V_{2,k}^n) = T\mu\sigma_k(V_{2,k}^n) \subseteq \operatorname{dom}\exp_{W_l}$. Thus we may consider:

$$\begin{aligned}
(\exp_{W_l}\sigma_l) \circ \exp_{W_l} T_x\lambda\sigma_k^{*_i}(x) &= \exp_{W_l}\sigma_l\mu(\exp_{W_k}\sigma_k|_{\Omega_{2,K_{5,i}}})^{-1}(z) \\
&= \exp_{W_k} T\mu\sigma_k(\exp_{W_k}\sigma_k|_{\Omega_{2,K_{5,i}}})^{-1}(z) \\
&= \mu(\exp_{W_k}\sigma_k)(\exp_{W_k}\sigma_k|_{\Omega_{2,K_{5,i}}})^{-1}(z) = \mu(z) = \lambda(z) \in \Omega_{\frac{5}{4},K_{5,j}}
\end{aligned}$$

Hence we deduce from (D.0.18): $\exp_{W_l} T\lambda\sigma_k^{*_i}(x) = \exp_{W_l}\sigma_l^{*_j}(\lambda(x))$. Since $\exp_{W_l}$ restricts to a diffeomorphism on $T_{\lambda(x)} \cap \hat{O}_j$, the computation yields (6.1.5). □

The families $\{\sigma_j \diamond_i \tau_j\}_{i\in I}$ and $\{\sigma_j^{*_i}\}_{i\in I}$ obtained in this way induce orbisections:

6.1.8 Proposition *Consider orbisections $[\hat{\sigma}],[\hat{\tau}] \in \mathcal{H}$, whose canonical families with respect to $\mathcal{B}$ are given by $\{\sigma_j | j \in J\}, \{\tau_j | j \in J\}$. Then*

(a) *The family $\left\{ \sigma_{\alpha(i)} \diamond_i \tau_{\alpha(i)} \middle| i \in I \right\}$ induces an orbisection $[\widehat{\sigma \diamond \tau}] \in \mathcal{M}$ whose family of canonical lifts with respect to the atlas $\mathcal{A}$ is given by $(\sigma \diamond \tau)_i := \sigma_{\alpha(i)} \diamond_i \tau_{\alpha(i)}|_{U_i}, i \in I$*

(b) *The family $\left\{ \sigma^{*_i}_{\alpha(i)} \middle| i \in I \right\}$ induces an orbisection $[\widehat{\sigma^*}] \in \mathcal{M}$ whose canonical lifts with respect to the atlas $\mathcal{A}$ are given by $(\sigma^*)_i := \sigma^{*_i}_{\alpha(i)}|_{U_i}, i \in I$*

Proof. The families $(\sigma \diamond \tau)_{i \in I}$ and $((\sigma^*)_i)_{i\in I}$ are compatible family of vector fields on the atlas $\mathcal{A}$ by Lemma 6.1.7. These families yield a canonical family of lifts with respect to the atlas $\mathcal{A}$. In particular the identities (6.1.4) and (6.1.5) allow the definition of continuous maps:

$$\sigma \diamond \tau \colon Q \to \mathcal{T}Q, x \mapsto T\psi_i(\sigma \diamond \tau)_i \psi_i^{-1}(x) \quad \text{ if } x \in \psi_i(U_i)$$
$$\sigma^* \colon Q \to \mathcal{T}Q, x \mapsto T\psi_i(\sigma^*)_i \psi_i^{-1}(x) \quad \text{ if } x \in \psi_i(U_i)$$

These data allow the definition of orbisections $[\widehat{\sigma \diamond \tau}]$ and $[\widehat{\sigma^*}]$ by remark 4.2.10 (a).
To complete the proof, we have to show that $[\widehat{\sigma \diamond \tau}], [\widehat{\sigma^*}]$ are contained in $\mathcal{M}$. To this end we need to assure that $[\widehat{\sigma \diamond \tau}]$ and $[\widehat{\sigma^*}]$ are compactly supported. The orbisections $[\hat{\sigma}],[\hat{\tau}] \in \mathcal{H}$ are compactly supported, whence $\mathrm{supp}[\hat{\sigma}] \cup \mathrm{supp}[\hat{\tau}]$ is contained in a compact subset $K \subseteq Q$. Since $\mathcal{B}$ is locally finite, there is only a finite subset $\mathcal{S}_{\sigma,\tau} \subseteq \mathcal{B}$ such that $(W_j, H_j, \varphi_j) \in \mathcal{S}_{\sigma,\tau}$ if and only if $\mathrm{Im}\,\varphi_j \cap K \neq \emptyset$. Consider $(W_j, H_j, \varphi_j) \in \mathcal{B} \setminus \mathcal{S}_{\sigma,\tau}$. By Remark 4.2.10 (d) the canonical lifts of $[\hat{\sigma}],[\hat{\tau}]$ on W_j are the zero-section in $\mathfrak{X}(W_j)$. The conclusion in Construction D.0.8 implies that $\sigma_j \diamond_i \tau_j \equiv 0$ and $\sigma_j^{*_i} \equiv 0$ hold for each $i \in \alpha^{-1}(j)$. Therefore the supports $\mathrm{supp}[\widehat{\sigma \diamond \tau}], \mathrm{supp}[\widehat{\sigma^*}]$ are contained in $K_{\sigma,\tau} := \bigcup_{(W_{\alpha(i)},H_{\alpha(i)},\varphi_{\alpha(i)}) \in \mathcal{S}_{\sigma,\tau}} \overline{\psi_i(U_i)}$. As $\mathcal{S}_{\sigma,\tau}$ is finite and for $j \in J$ the set $\alpha^{-1}(j)$ is finite, $K_{\sigma,\tau}$ is a finite union of compact sets $\overline{\psi_i(U_i)}$. Hence the supports of $[\widehat{\sigma \diamond \tau}]$ respectively $[\widehat{\sigma^*}]$ are contained in a compact set, whecne these orbisections are compactly supported.
Following Proposition 4.2.9 we may consider the canonical lifts $(\sigma \diamond \tau)_k$ and σ_k^* on each chart $(W_k, H_k, \varphi_k) \in \mathcal{B}$. The orbisections $[\widehat{\sigma \diamond \tau}], [\widehat{\sigma^*}]$ will be contained in $\mathcal{M}$ if their respective canonical lifts are contained in $\mathcal{M}_i$ for each $i \in \alpha^{-1}(k)$, $k \in J$. Fix $i \in \alpha^{-1}(k)$ and denote by $(\sigma \diamond \tau)_k)_{[n]}$ respectively $(\sigma_k^*)_{[n]}$ the local representatives of the lifts in a chart $(V^n_{5,k}, \kappa^n_k) \in \mathcal{F}_5(K_{5,i})$. By construction 6.1.6 it suffices to prove that for each chart in $\mathcal{F}_5(K_{5,i})$ the condition $\left\|((\sigma \diamond \tau)_k)_{[n]}\right\|_{\overline{B_1(0)},1} < \tau_i$ resp. $\left\|(\sigma_k^*)_{[n]}\right\|_{\overline{B_1(0)},1} < \tau_i$ holds. Observe that this condition may be checked on $\Omega_{\frac{5}{4},K_{5,i}}$. Uniqueness of canonical lifts together with (6.1.4) and (6.1.5), forces the canonical lifts $(\sigma\diamond\tau)_k$ respectively $(\sigma^*)_k$ to coincide with $\sigma_k \diamond_i \tau_k$ respectively $\sigma_k^{*_i}$ on $\Omega_{\frac{5}{4},K_{5,i}}$. Recall from the construction that the constant τ_i corresponds to the constant τ in Construction D.0.8. Hence a combination of (D.0.16) with Corollary D.0.9 yields $\left\|((\sigma \diamond \tau)_k)_{[n]}\right\|_{\overline{B_1(0)},1} = \|\sigma_k \diamond_n \tau_k\|_{\overline{B_1(0)},1} < \tau_i$ and $\left\|(\sigma_k^*)_{[n]}\right\|_{\overline{B_1(0)},1} = \|\sigma_k^{*_n}\|_{\overline{B_1(0)},1} < \tau_i$. We conclude that each of the canonical lifts of $[\widehat{\sigma \diamond \tau}]$ and $[\widehat{\sigma^*}]$ on (W_k, H_k, φ_k) is contained in $\mathcal{M}_i$ with $i \in \alpha^{-1}(k)$. Summing up, $[\widehat{\sigma \diamond \tau}]$ and $[\widehat{\sigma^*}]$ are contained in $\mathcal{M}$. ☐

The last Lemma implies that the map E may be applied to $[\sigma \diamond \tau], [\sigma^*]$ for $[\sigma],[\tau] \in \mathcal{H}$. We shall now assure that the orbisections constructed, satisfy the identities needed for composition and inversion in $E(\mathcal{M})$:

6.1.9 Lemma *Consider $[\hat{\sigma}], [\hat{\tau}] \in \mathcal{H}$. The following identities hold:*

$$E([\hat{\sigma}]) \circ E([\hat{\tau}]) = E([\widehat{\sigma \diamond \tau}]) \tag{6.1.6}$$

$$E([\hat{\sigma}])^{-1} = E([\widehat{\sigma^*}]) \tag{6.1.7}$$

Proof. Choose and fix arbitrary $[\hat{\sigma}], [\hat{\tau}] \in \mathcal{H}$. The left hand and the right hand sides of the equations (6.1.6) resp. (6.1.7) are orbifold diffeomorphisms. As observed in Corollary 3.1.11, orbifold diffeomorphisms are uniquely determined by their families of lifts. To prove the assertion, we need to find orbifold atlases such that the families of local lifts for both sides of the equations coincide for these atlases.
Consider at first the right hand sides of both equations: The orbisections $[\sigma \diamond \tau]$ and $[\sigma^*]$ have been constructed by a family of canonical lifts $\{ (\sigma \diamond \tau)_i | i \in I \}$ resp. $\{ (\sigma^*)_i | i \in I \}$ with respect to the atlases $\mathcal{A}$ and $\mathcal{TA}$. Both orbisections are contained in $\mathcal{M}$. Taking identifications $\operatorname{Im}(\sigma \diamond \tau)_i, \operatorname{Im}(\sigma^*)_i \subseteq \hat{O}_{\alpha(i)}$ holds. Corestriction of each lift to $TU_i \cap \hat{O}_{\alpha(i)}$ yields representatives of $[\hat{\sigma}]|^{\Omega}$ and $[\hat{\tau}]|^{\Omega}$. Thus representatives of $E([\widehat{\sigma \diamond \tau}])$ and $E([\widehat{\sigma^*}])$ are given by $(E^{\sigma \diamond \tau}, \{ e^{(\sigma \diamond \tau)_i} \}_{i \in I}, P, \nu)$ respectively $(E^{\sigma^*}, \{ e^{\sigma_i^*} \}_{i \in I}, P', \nu')$ in $\operatorname{Orb}(\mathcal{A}, \mathcal{C})$. The lifts of these maps satisfy for each $i \in I$ by construction:

$$\exp_{W_{\alpha(i)}} \circ \sigma_{\alpha(i)} \circ \exp_{W_{\alpha(i)}} \circ \tau_i = \exp_{W_{\alpha(i)}} \circ (\sigma \diamond \tau)_i = e^{(\sigma \diamond \tau)_i} \tag{6.1.8}$$

$$(\exp_{W_{\alpha(i)}} \circ \sigma_{\alpha(i)}|_{\Omega_{2,i}})^{-1}|_{U_i} = \exp_{W_{\alpha(i)}} \circ \sigma_i^* = e^{\sigma_i^*} \tag{6.1.9}$$

As **Orb** is a category, composition in **Orb** is associative. Hence lifts may be computed iteratively: $E([\sigma]) \circ E([\tau]) = [\exp_{\mathrm{Orb}}] \circ [\sigma]|^{\Omega} \circ [\exp_{\mathrm{Orb}}] \circ [\tau]|^{\Omega} = [\exp_{\mathrm{Orb}}] \circ ([\sigma]|^{\Omega} \circ [\exp_{\mathrm{Orb}}] \circ [\tau]|^{\Omega})$.As $\tau_{\alpha(i)}, \sigma_{\alpha(i)}$ are contained in $\mathcal{H}_{R_i}$, the composition of charted orbifold maps (cf. Construction E.4.1) yields a lift of $E^{\sigma} \circ E^{\tau}$ on U_i which coincide with the left hand side of (6.1.8). Therefore (6.1.6) follows from (6.1.8) by an application of Corollary 3.1.11.
Consider the representative $\hat{E}^{\sigma}$ of $[\hat{E}^{\sigma}]$ constructed in Proposition 6.1.4. Corollary 3.1.12 implies that the inverse $E([\sigma])^{-1}$ is induced by a representative $(\hat{E}^{\sigma})^{-1} := ((E^{\sigma})^{-1}, \{ (e^{\sigma_i})^{-1} | i \in I \}, R, \nu'')$. Here the lifts are defined via $(e^{\sigma_i})^{-1} := (\exp_{W_{\alpha(i)}} \circ \sigma_i)^{-1} \colon \operatorname{Im} e^{\sigma_i} \to U_i, \ i \in I$.
We will construct charts such that the families $\{ (e^{\sigma_i})^{-1} | i \in I \}$ and $\{ e^{\sigma_i^*} | i \in I \}$ induce the same lifts on these charts. To this end fix $i \in I$ and $y \in U_i$. Then there is a manifold chart $(V_{5,\alpha(i)}^n, \kappa_n^{\alpha(i)}) \in \mathcal{F}_5(H_{\alpha(i)}.\overline{U_i})$ such that $y \in V_{1,\alpha(i)}^n$ holds. The Furthermore each chart in $\mathcal{F}_5(H_{\alpha(i)}.U_i)$, is contained in $\Omega_{1,K_{5,i}}$ by definition (cf. Construction 6.1.6 and Construction 6.1.1 IV.). The neighborhood $\mathcal{H}_{R_i}$ has been constructed by an application of Construction D.0.8 with respect to the family $\mathcal{F}_5(K_{5,i})$. Since $(V_{5,\alpha(i)}^n, \kappa_n^{\alpha(i)}) \in \mathcal{F}_5(H_{\alpha(i)}.\overline{U_i})$ and $\sigma_{\alpha(i)} \in \mathcal{H}_{R_i}$ as $[\hat{\sigma}] \in \mathcal{H}$ holds, property ii. in the proof of Lemma 6.1.7 yields $x := e^{\sigma_{\alpha(i)}^*}(y) \in V_{2,\alpha(i)}^n \subseteq \Omega_{2,K_{5,i}}$. Observe that The family $\{ (U_i, G_i, \psi_i) \}_{i \in I}$ is an orbifold atlas, whence there is some $j \in I$, $z \in U_j$ with $\psi_j(z) = \psi_i(x)$. Then $x \in \mathcal{K}_{\alpha(i)}$ and $z \in \mathcal{K}_{\alpha(j)}$ hold. Argueing as in the proof of Lemma 6.1.8, there is a change of charts map $\mu \colon W_{\alpha(i)} \supseteq \operatorname{dom} \mu \to W_{\alpha(j)}$, such that $\mu(x) = z$, $V_{5,\alpha(i)}^n \subseteq \operatorname{dom} \mu$ are satisfied. Analogously to the proof of Lemma 6.1.7, $\sigma_{\alpha(i)} \in \mathcal{H}_{R_i}$ implies via a local computation: $\exp_{W_{\alpha(i)}} \circ \sigma_{\alpha(i)}(V_{2,\alpha(i)}^n) \subseteq V_{5,\alpha(i)}^n \subseteq \operatorname{dom} \mu$ and

$$\mu \exp_{W_{\alpha(i)}} \sigma_{\alpha(i)|V_{2,\alpha(i)}^n} = \exp_{W_{\alpha(j)}} T\mu \sigma_{\alpha(i)|V_{2,\alpha(i)}^n} = \exp_{W_{\alpha(j)}} \sigma_{\alpha(j)} \mu_{|V_{2,\alpha(i)}^n}. \tag{6.1.10}$$

Choose a $H_{\alpha(i)}$-stable neighborhood $x \in S_x \subseteq V^n_{2,\alpha(i)} \cap e^{\sigma_i^*}(V^n_{1,\alpha(i)}) \cap \mu^{-1}(U_j)$. Then μ restricts to an embedding μ' of orbifold charts mapping the chart $(S_x, H_{\alpha(i),S_x}, \varphi_{\alpha(i)}|_{S_x})$ into (U_j, G_j, ψ_j). The set $\mu'(S_x)$ is a G_j-stable subset of U_j by [48, Proposition 2.12 (i)] and as such $H_{\alpha(j)}$-stable. The orbisection $[\hat{\sigma}]$ is an element of $\mathcal{H} \subseteq \mathcal{M}$ and by construction $H_{\alpha(j)}.\mu(S_x) \subseteq H_{\alpha(j)}.U_j \subseteq \Omega_{2,j}$ holds. Analogous to Steps 2 and 3 in the proof of Lemma 6.1.2, the set $e^{\sigma_j}(\mu(S_x))$ is $H_{\alpha(i)}$-stable. From (6.1.10) we derive $e^{\sigma_j}(\mu(S_x)) \subseteq \operatorname{cod}\mu$. The change of charts μ^{-1} restricts to an open embedding of orbifold charts on the stable set $e^{\sigma_j}(\mu(S_x))$. Hence $S_y := \mu^{-1}e^{\sigma_j}(\mu(S_x))$ is a $H_{\alpha(i)}$-stable subset. Consider $z \in S_y$ and let y_z be the unique element in $V^n_{1,\alpha(i)}$ with $z = \mu^{-1}e^{\sigma_j}(\mu(e^{\sigma_i^*}(y_z))$. From property ii. in the proof of Lemma 6.1.7 with (6.1.10) we deduce $z = \mu^{-1}\mu e^{\sigma_i}e^{\sigma_i^*}(y_z) = y_z \in V^n_{1,\alpha(i)}$. The computation yields $y \in S_y \subseteq V^n_{1,\alpha(i)}$. So far we have achieved that $\mu'' := \mu|_{S_y}$ is an embedding of orbifold charts such that $e^{\sigma_j}\mu' = \mu'' \exp_{W_{\alpha(i)}} \circ \sigma_i|_{S_x}$. The lift e^{σ_j} and the mapping $\exp_{W_{\alpha(i)}} \circ \sigma_i|_{V^n_{2,\alpha(i)}}$ are open embeddings with $e^{\sigma_i}(S_x) = S_y$ and $e^{\sigma_j}\mu(S_x) = \mu(S_y)$. Hence we derive

$$(e^{\sigma_j})^{-1}\mu'' = \mu'(\exp_{W_{\alpha(i)}} \circ \sigma_i|_{S_x})^{-1} = \mu' e^{\sigma_i^*}|_{S_y}$$

The last identity follow from $S_x \subseteq V^n_{2,\alpha(i)} \subseteq \Omega_{\frac{5}{4},K_{5,i}}$ by (D.0.18). Summing up, $E(\hat{\sigma})^{-1}$ and $E(\widehat{[\sigma^*]})$ induce the same lift with respect to $(S_y, H_{\alpha(i),S_y}, \varphi_{\alpha(i)}|_{S_y})$ and $(S_x, H_{\alpha(i),S_x}, \varphi_{\alpha(i)}|_{S_x})$. As $i \in I$ and $y \in U_i$ were arbitrary, we may construct an orbifold atlas of $(Q,\mathcal{U})$, such that the lifts of both orbifold diffeomorphisms coincide with respect to this atlas. □

We now turn our attention to the composition and inversion maps:

6.1.10 Lemma *The maps*

$$\begin{aligned} \operatorname{comp}\colon \mathcal{H} \times \mathcal{H} \to \mathcal{M} \subseteq \mathfrak{X}_{Orb}(Q)_c\,, &\quad ([\hat{\sigma}],[\hat{\tau}]) \mapsto [\widehat{\sigma \diamond \tau}] \\ \operatorname{inv}\colon \mathcal{H} \to \mathcal{M} \subseteq \mathfrak{X}_{Orb}(Q)_c\,, &\quad [\hat{\sigma}] \mapsto [\widehat{\sigma^*}] \end{aligned}$$

are smooth.

Proof. The atlases $\mathcal{A}$ and $\mathcal{C}$ are indexed by I. Let σ_i respectively $\sigma_{\alpha(i)}$ be the canonical lifts with respect to $(U_i, G_i, \psi_i) \in \mathcal{A}$ respectively $(W_{\alpha(i)}, H_{\alpha(i)}, \varphi_{\alpha(i)}) \in \mathcal{C}$. The continuous linear maps $\tau_i\colon \mathfrak{X}_{\mathrm{Orb}}(Q)_c \to \mathfrak{X}(U_i), [\hat{\sigma}] \mapsto \sigma_i$ and $\lambda_i\colon \mathfrak{X}_{\mathrm{Orb}}(Q)_c \to \mathfrak{X}(W_{\alpha(i)}), [\hat{\sigma}] \mapsto \sigma_{\alpha(i)}$ induce patchworks for $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$ by Corollary 4.3.6. The product $\mathfrak{X}_{\mathrm{Orb}}(Q)_c \times \mathfrak{X}_{\mathrm{Orb}}(Q)_c$ is a locally convex vector space together with a family of maps $\lambda_i \times \lambda_i\colon \mathfrak{X}_{\mathrm{Orb}}(Q)_c \times \mathfrak{X}_{\mathrm{Orb}}(Q)_c \to \mathfrak{X}(W_{\alpha(i)}) \times \mathfrak{X}(W_{\alpha(i)})$, $i \in I$. Arguments as in the proof of Lemma D.0.10 show that the family $(\lambda_i \times \lambda_i)_{i\in I}$ yields a patchwork for $\mathfrak{X}_{\mathrm{Orb}}(Q)_c \times \mathfrak{X}_{\mathrm{Orb}}(Q)_c$. Let p be the correpsonding topological embedding for this patched space (cf. Definition C.3.4).
The patchwork on each of the spaces $(\mathfrak{X}_{\mathrm{Orb}}(Q)_c \times \mathfrak{X}_{\mathrm{Orb}}(Q)_c, (\lambda_i \times \lambda_i)_{i\in I})$, $(\mathfrak{X}_{\mathrm{Orb}}(Q)_c, (\lambda_i)_{i\in I})$ and $(\mathfrak{X}_{\mathrm{Orb}}(Q)_c, (\tau_i)_{i\in I})$, is indexed by I. On the open set $\mathcal{H}_{R_i}$ constructed in 6.1.6 consider the maps

$$\begin{aligned} \operatorname{comp}_i\colon \mathcal{H}_{R_i} \times \mathcal{H}_{R_i} &\to \mathfrak{X}(U_i), (X,Y) \mapsto X \diamond_i Y|_{U_i} \\ \operatorname{inv}_i\colon \mathcal{H}_{R_i} &\to \mathfrak{X}(U_i), X \mapsto X^{*_i}|_{U_i}. \end{aligned}$$

Since $\mathcal{H} = \Lambda_{\mathcal{C}}^{-1}(\oplus_{i\in I}\mathcal{H}_{R_i})$ holds, the identities for the patchwork established in the proof of Lemma D.0.10 yield $p(\mathcal{H}\times\mathcal{H}) \subseteq \oplus_{i\in I}(\mathcal{H}_{R_i}\times\mathcal{H}_{R_i})$ and $\Lambda_{\mathcal{C}}(\mathcal{H}) \subseteq \oplus_{i\in I}\mathcal{H}_{R_i}$. By construction, we deduce from Proposition 6.1.8:

$$(\mathrm{comp}_i)_{i\in I}\, p|_{\mathcal{H}\times\mathcal{H}}^{\oplus(\mathcal{H}_{R_i}\times\mathcal{H}_{R_i})} = \Lambda_{\mathcal{A}}\circ\mathrm{comp} \text{ and } (\mathrm{inv}_i)_{i\in I}\,\Lambda_{\mathcal{C}}|_{\mathcal{H}}^{\oplus_{i\in I}\mathcal{H}_{R_i}} = \Lambda_{\mathcal{A}}\,\mathrm{inv}\,.$$

These mappings are well-defined, since comp_i and inv_i vanish on the zero-element. Hence comp and inv are patched mappings. By Proposition C.3.7 it is sufficient to prove that comp and inv are smooth on the patches, i.e. for each $i \in I$ the maps comp_i and inv_i are smooth. For the remainder of this proof we therefore fix $i \in I$ and prove the smoothness of comp_i respectively inv_i:
The open sets $\Omega_{r,K_{5,i}}$, $r \in [1,5]$ contain U_i. Consider the restriction maps $\mathrm{res}_{\Omega_{5,K_{5,i}}}^{W_{\alpha(i)}}$, $\mathrm{res}_{U_i}^{\Omega_{r,K_{5,i}}}$ which are linear and continuous, whence smooth by [25, Lemma F.15 (a)]. Recall that the maps

$$\begin{aligned} c_i &\colon \mathcal{H}_{R_i}^{\Omega_{5,K_{5,i}}} \times \mathcal{H}_{R_i}^{\Omega_{5,K_{5,i}}} \to \mathfrak{X}\left(\Omega_{2,K_{5,i}}\right), X \mapsto X \diamond_i Y \\ \iota_i &\colon \mathcal{H}_{R_i}^{\Omega_{5,K_{5,i}}} \to \mathfrak{X}\left(\Omega_{\frac{5}{4},K_{5,i}}\right), X \mapsto X^{*_i} \end{aligned}$$

are smooth by Lemma D.0.10. By definition the maps comp_i and inv_i are given as compositions:

$$\begin{aligned} \mathrm{comp}_i &= \mathrm{res}_{U_i}^{\Omega_{2,K_{5,i}}} \circ c_i \circ \mathrm{res}_{\Omega_{5,K_i}}^{W_{\alpha(i)}} \times \mathrm{res}_{\Omega_{5,K_i}}^{W_{\alpha(i)}} |_{\mathcal{H}_{R_i}\times\mathcal{H}_{R_i}} \\ \mathrm{inv}_i &= \mathrm{res}_{U_i}^{\Omega_{\frac{5}{4},K_{5,i}}} \circ \iota_i \circ \mathrm{res}_{\Omega_{5,K_i}}^{W_{\alpha(i)}} |_{\mathcal{H}_{R_i}} \end{aligned}$$

We conclude that comp_i and inv_i are smooth, whence comp and inv are smooth. □

We are now in a position to construct a Lie group structure on a subgroup of $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$:

6.1.11 Proposition *There is a subset $\mathcal{P} \subseteq \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ which contains the identity, such that the subgroup generated by $\mathcal{P}$*

$$\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0 := \langle\mathcal{P}\rangle$$

admits a unique smooth manifold structure, turning $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$ into a connected Lie group and $\mathcal{P}$ into an open connected identity-neighborhood.

Proof. Endow $E(\mathcal{M})$ with the unique smooth manifold structure turning $E\colon \mathcal{M} \to E(\mathcal{M})$ into a diffeomorphism. Consider $\mathcal{P}_0 := E(\mathcal{H})$ as an open submanifold of $E(\mathcal{M})$. Combining Lemma 6.1.9 and Lemma 6.1.10 the composition and inversion

$$\begin{aligned} &m\colon \mathcal{P}_0\times\mathcal{P}_0 \to E(\mathcal{M}), ([\hat{f}],[\hat{g}]) \mapsto [\hat{f}]\circ[\hat{g}] = E(\mathrm{comp}(E^{-1}([\hat{f}]),E^{-1}([\hat{g}]))) \\ &\iota\colon \mathcal{P}_0 \to E(\mathcal{M}), [\hat{f}] \mapsto [\hat{f}]^{-1} = E(\mathrm{inv}(E^{-1}([\hat{f}])) \end{aligned}$$

are smooth maps. Observe that by Proposition 6.1.8 and definition of m and ι the images are contained in $E(\mathcal{M})$. The set $\mathcal{P}_0$ is an open identity-neighborhood on which inversion and group multiplication of $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ are smooth. Hence the inverse image $\iota^{-1}(\mathcal{P}_0) = \mathcal{P}_0 \cap (\mathcal{P}_0)^{-1}$, with

$(\mathcal{P}_0)^{-1} := \iota(\mathcal{P}_0)$ is an open neighborhood of the identity in $\mathcal{P}_0$. Thus $E^{-1}(\mathcal{P}_0 \cap (\mathcal{P}_0)^{-1})$ is an open zero-neighborhood in $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$. Since this space is locally convex, we may choose a convex zero neighborhood $\mathcal{H}_1 \subseteq E^{-1}(\mathcal{P}_0 \cap (\mathcal{P}_0)^{-1}) \subseteq \mathfrak{X}_{\mathrm{Orb}}(Q)_c$. Then $\mathcal{P}_1 := E(\mathcal{H}_1) \subseteq \mathcal{P}_0 \cap (\mathcal{P}_0)^{-1}$ is a connected, open identity neighborhood in $E(\mathcal{M})$. Since $\mathcal{P}_1 \subseteq \mathcal{P}_0 \cap (\mathcal{P}_0)^{-1}$ holds, we have $\iota^{-1}(\mathcal{P}_1) = \mathcal{P}_0 \cap (\mathcal{P}_1)^{-1} = (\mathcal{P}_1)^{-1} = \iota(\mathcal{P}_1)$. Being an inverse image of an open set with respect to a continuous map, $(\mathcal{P}_1)^{-1}$ is open. Furthermore it is connected as continuous image of such a set. We obtain an open, connected identity-neighborhood $\mathcal{P} := \mathcal{P}_1 \cup (\mathcal{P}_1)^{-1} \subseteq \mathcal{P}_0$ is in $E(\mathcal{M})$ by [20, Corollary 6.1.10].
From the above we deduce $m(\mathcal{P},\mathcal{P}) \subseteq E(\mathcal{M})$ and the mapping $\mathcal{P} \times \mathcal{P} \to E(\mathcal{M}), ([\hat{f}],[\hat{g}]) \mapsto [\hat{f}] \circ [\hat{g}]$ induced by m is a smooth map. Furthermore $(\mathcal{P})^{-1} = \mathcal{P} \subseteq E(\mathcal{M})$ holds and the mapping $\mathcal{P} \to E(\mathcal{M}), [\hat{f}] \mapsto [\hat{f}]^{-1}$ induced by ι is smooth. In conclusion all prerequesits of Proposition C.4.3 (a) have been checked. Hence we derive a unique smooth manifold structure on

$$\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0 := \langle \mathcal{P} \rangle$$

turning it into a Lie group, such that $\mathcal{P}$ is an open identity-neighborhood in $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$. In addition the manifold structure induced by $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$ coincides with the submanifold structure of $\mathcal{P} \subseteq E(\mathcal{M})$. Therfore $\mathcal{P} \subseteq \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$ is open and connected. As the the group operations of $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$ are smooth, each of the sets $\mathcal{P}^n$ (the elements of $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$, which are obtained by n-fold composition of elements in $\mathcal{P}$, $n \in \mathbb{N}$) is a connected identity-neighborhood. Since $\mathcal{P}$ is a symmetric identity-neighborhood, we deduce from the proof of [35, Theorem 5.7]:

$$\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0 = \langle \mathcal{P} \rangle = \bigcup_{n=1}^{\infty} \mathcal{P}^n.$$

Hence $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$ is a connected Lie group by [20, Corollary 6.1.10]. □

In the next section we shall construct a Lie group structure on $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$. The Lie group structure on the subgroup $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$ of $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ will turn this subgroup into the identity component for the Lie group $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$.

6.2. Lie group structure on $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$

Unless stated otherwise all symbols used in this section retain the same meaning as in Section 6.1. In particular we shall always be working with a Riemannian orbifold $(Q,\mathcal{U},\rho)$. At first we will prove that the Lie group $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$ is independent of the choice of the atlases $\mathcal{A},\mathcal{B}$ and the local data constructed in Section 6.1. Secondly the construction does not depend on the choice of the Riemannian orbifold metric on $(Q,\mathcal{U})$. Having dealt with these preparations, an application of the Construction Principle C.4.3, yields a unique smooth Lie group structure on $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$.

6.2.1 Lemma *The Lie group $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$ constructed in Proposition 6.1.11 does neither depend on the choice of atlases $\mathcal{A}$ and $\mathcal{B}$, nor on the local data collected in Construction 6.1.1.*

Proof. Let $\mathcal{A}^+$ and $\mathcal{B}^+$ be orbifold atlases, which satisfy the same properties as $\mathcal{A}$ and $\mathcal{B}$ in Construction 6.1.1. Replace $\mathcal{A}$ and $\mathcal{B}$ in the construction constructions of Section 6.1 with $\mathcal{A}^+$ and $\mathcal{B}^+$. Taking the Riemannian orbifold metric ρ as before, we obtain another connected, smooth Lie group $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0^+$ depending on the new set of data. As shown in Section 6.1 there is a C^∞-diffeomorphism E^+, $E^+([\hat{\sigma}]) := [\exp_{\mathrm{Orb}}] \circ [\hat{\sigma}]$ mapping the open convex zero-neighborhood $\mathcal{H}_1^+$ (defined as in Proposition 6.1.11 with repsect to $\mathcal{A}^+$ and $\mathcal{B}^+$, the open subset $\mathcal{H}^+ \subseteq \mathfrak{X}_{\mathrm{Orb}}(Q)_c$ and the local data constructed for $\mathcal{A}^+$, $\mathcal{B}^+$) onto an open identity neighborhood in $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0^+$. Then $O := \mathcal{H}_1 \cap \mathcal{H}_1^+$ is an open, convex (and hence connected) zero-neighborhood in $\mathfrak{X}_{\mathrm{Orb}}(Q,\mathcal{U})_c$. The map E takes O diffeomorphically onto an open identity neighborhood in $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$. As $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$ is a connected Lie group, $E(O)$ generates this group by [35, Theorem 7.4]. Analogously E^+ maps O diffeomorphically onto an open identity neighborhood in $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0^+$ which generates this group. Recall from Proposition 6.1.5 that $E([\hat{\sigma}]) = [\exp_{\mathrm{Orb}}] \circ [\hat{\sigma}]|^{\Omega} = E^+([\hat{\sigma}])$ holds for each $[\hat{\sigma}] \in O$. Hence both maps coincide on O. We deduce $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0 = \langle E(O) \rangle = \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0^+$ as an abstract group, and also as a Lie group. □

6.2.2 Lemma *The Lie group* $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$ *constructed in Proposition 6.1.11 does not depend on the choice of Riemannian orbifold metric* ρ *on* $(Q,\mathcal{U})$ *(cf. Section 6.1).*

Proof. Let $\rho^\#$ be another Riemannian orbifold metric on $(Q,\mathcal{U})$. By Lemma 6.2.1 we may use the same atlases $\mathcal{A} := \{ (U_i, G_i, \psi_i) | i \in I \}$ and $\mathcal{B} := \{ (W_j, H_j, \varphi_j) | j \in J \}$ as in Construction 6.1.1. Reviewing this, the local data constructed in Construction 6.1.1 II. - IV. does **not depend** on the Riemannian orbifold metric. The constants $R_i,\ i \in I$ and $s_j, S_j,\ j \in J$ in Construction 6.1.1 V. change for $\rho^\# = (\rho_j^\#)_{j \in J}$. The new constants depending on $\rho^\#$ will be denoted by $R_i^\#,\ i \in I$ and $s_j^\#, S_j^\#, j \in J$ (see Construction 6.1.1 V. for their properties).
Let $[\widehat{\exp_{\mathrm{Orb}}}^\#]$ be the Riemannian orbifold exponential map with respect to $(Q, \mathcal{U}, \rho^\#)$. As in Section 6.1, one constructs open zero-neighborhoods $\mathcal{H}^\# := \Lambda_{\mathcal{C}}^{-1}(\oplus_{i \in I} \mathcal{H}_{R_i^\#})$ and $\mathcal{H}^\# \subseteq \mathcal{M}^\#$, which depend on the data in Construction 6.1.1 I. - IV., the constants $R_i^\#,\ i \in I$ and $s_j^\#, S_j^\#, j \in J$, as well as on the Riemannian orbifold metric $\rho^\#$. Furthermore we obtain an injective map $E^\# \colon \mathcal{M}^\# \to \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$, a connected Lie group $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0^\# = \langle \mathcal{P}^\# \rangle$ and a convex zero-neighborhood $\mathcal{H}_0^\# \subseteq \mathcal{H}^\# \subseteq \mathfrak{X}_{\mathrm{Orb}}(Q)_c$, such that $E^\#|_{\mathcal{H}_0^\#} \colon \mathcal{H}_0^\# \to \mathcal{P}^\# \subseteq \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0^\#, [\hat{\sigma}] \mapsto [\widehat{\exp_{\mathrm{Orb}}}^\#] \circ [\hat{\sigma}]|^{\Omega^\#}$ is a diffeomorphism onto an open identity neighborhood.
Fix some $i \in I$ and let $\mathcal{F}_5(K_{5,i}) := \left\{ (V_{5,\alpha(i)}^n, \kappa_n^{\alpha(i)}) \middle| 1 \leq n \leq N_i \right\}$ be the atlas of Construction 6.1.1 IV.[4]. For each $1 \leq n \leq N_i$ the Riemannian metrics induce pullback metrics with respect to the manifold charts $\kappa_n^{\alpha(i)}$. The charts $\kappa_n^{\alpha(i)}$ induce pullback metrics on $B_5(0)$ with respect to $\rho_{\alpha(i)}$ and $\rho_{\alpha(i)}^\#$. For $(V_{5,\alpha(i)}^n, \kappa_n^{\alpha(i)}), 1 \leq n \leq N_i$. The associated Riemannian exponential maps will be denoted by $\exp_{W_{\alpha(i)},[n]}$ respectively by $\exp_{W_{\alpha(i)},[n]}^\#$. Finally we define the local representatives of $X \in \mathfrak{X}\left(W_{\alpha(i)}\right)$ with respect to $\kappa_n^{\alpha(i)}$ via $X_{[n]} := X_{\kappa_n^{\alpha(i)}} \circ (\kappa_n^{\alpha(i)})^{-1} \in C^\infty(B_5(0), \mathbb{R}^d)$.

[4] To shorten our notation we number all charts from 1 to some $N_i \in \mathbb{N}$, $i \in I$. From the constext it will be always clear which charts are meant.

Observe that the open set $\mathcal{H}_{R_i}$ in Construction 6.1.6 was obtained by Construction D.0.8. Reviewing this construction, for $1 \leq n \leq N_i$, $\varepsilon_n, \delta_n > 0$ have been chosen, such that for each $x \in \overline{B_4(0)}$ the map $\phi_{\alpha(i),[n],x} \colon B_{\varepsilon_n}(0) \to B_{\delta_n}(x), y \mapsto \exp_{\alpha(i),[n]}(x,y)$ is a diffeomorphism. Furthermore the choice of ε_n yields the smooth map $b_{\alpha(i),[n]} \colon W_{\delta_n} \to B_{\varepsilon_n}(0), b(x,y) := \phi^{-1}_{\alpha(i),[n],x}(y)$ (cf. Lemma D.0.3). Recall that $\varepsilon_n < \nu_i$ holds for $1 \leq n \leq N_i$. Here ν_i is the constant constructed in Lemma D.0.6 with respect to the finite family $\mathcal{F}_5(K_{5,i})$. Thus the assertions of Lemma D.0.6 hold. For each $x \in V^n_{4,\alpha(i)}, 1 \leq n \leq N_i$, there is an open set $N_x \subseteq T_x W_{\alpha(i)}$, with the following property:

$$B_{\delta_n}(\kappa_n^{\alpha(i)}(x)) \subseteq \exp_{W_{\alpha(i)},[n]}(\kappa_n^{\alpha(i)}(x), B_{\varepsilon_n}(0)) \subseteq \kappa_n^{\alpha(i)} \exp_{W_{\alpha(i)}}(N_x). \tag{6.2.1}$$

Observe that the neighborhood $\mathcal{H}_{R_i^\#}$ has been obtained by another application of Construction D.0.8 with respect to a family of constants $\varepsilon_n^\#, \delta_n^\# > 0, 1 \leq n \leq N_i$.
By Lemma D.0.3 (c) we may choose constants $\varepsilon_n^\# > \varepsilon_{1,n}^\# > 0$ for $1 \leq n \leq N_i$ so small that $\exp^\#_{\alpha(i),[n]}\left(\left\{ \kappa_n^{\alpha(i)}(x) \right\} \times B_{\varepsilon^\#_{1,n}}(0)\right)$ is contained in $B_{\delta_n}(\kappa_n^{\alpha(i)}(x))$ for $x \in \overline{V^n_{4,\alpha(i)}}$. For $1 \leq n \leq N_i$ we choose for each $\varepsilon^\#_{1,n}$ a constant $\delta_n > \delta_{1,n} > 0$ which satisfies the assertion of Lemma D.0.3 (b). Apply Construction D.0.8 with $R := R_i^\#$ and $P := P^\#_{1,i} \cap P^\#_{2,i}$, but replace the pairs $(\varepsilon_n^\#, \delta_n^\#)$ with $(\varepsilon^\#_{1,n}, \delta^\#_{1,n})$ to obtain an open zero-neighborhood $H_{R_i^\#} \subseteq \mathcal{H}^{\Omega_{5,K_{5,i}}}_{R_i^\#}$. Thus the map

$$u_n \colon B_4(0) \times B_{\varepsilon^\#_{1,n}}(0) \to B_{\varepsilon_n}(0), u_n(x,y) := b_{\alpha(i),[n]}(x, \exp^\#_{\alpha(i),[n]}(x,y)) \tag{6.2.2}$$

is well-defined and smooth, as a composition of smooth maps. By construction $\varepsilon^\#_{1,n} < \varepsilon^\#_n < \nu^\#$ holds, where $\nu^\#$ is the constant as in Lemma D.0.6 with respect to the finite family $\mathcal{F}_5(K_{5,i}$. Hence we deduce with Lemma D.0.6 (b) from equations (6.2.2) and (6.2.1) that the map

$$(E^{-1}E^\#)_i \colon H_{R_i^\#} \to \mathfrak{X}\left(\Omega_{1,K_{5,i}}\right), (E^{-1}E^\#)_i(X) := \exp_{\alpha(i)}|^{-1}_{N_x} \exp^\#_{\alpha(i)} \circ X \tag{6.2.3}$$

is well-defined. In addition, we claim that $(E^{-1}E^\#)_i$ is a smooth map. To see this, let $1 \leq n \leq N_i$ and recall that $H_{R_i^\#} \subseteq \mathfrak{X}\left(\Omega_{5,K_{5,i}}\right)$ is open and $\mathcal{F}_5(K_{5,i})$ covers $\Omega_{5,K_{5,i}}$. Hence for $1 \leq n \leq N_i$ the maps $r_n \colon \mathfrak{X}\left(\Omega_{5,K_{5,i}}\right) \to C^\infty(B_5(0), \mathbb{R}^d), X \mapsto X_{[n]}$ form a patchwork by Definition C.3.1. Analogously the maps $t_n \colon \mathfrak{X}\left(\Omega_{1,K_{5,i}}\right) \to C^\infty(B_1(0), \mathbb{R}^d), X \mapsto X_{[n]}|_{B_1(0)}, \leq n \leq N_i$ yield a patchwork. Consider the open subset $\lfloor \overline{B_1(0)}, B_{\varepsilon^\#_{1,n}}(0) \rfloor_\infty \subseteq C^\infty(B_5(0), \mathbb{R}^d)$. For $X \in H_{R_i^\#}$ we obtain $X_{[n]}(\overline{B_3(0)}) \subseteq B_{\varepsilon^\#_{1,n}}(0)$ (cf. Construction D.0.8 and Lemma D.0.4). Hence $r_n(H_{R_i^\#}) \subseteq \lfloor \overline{B_1(0)}, B_{\varepsilon^\#_{1,n}}(0) \rfloor_\infty$ holds. In addition [25, Proposition 4.23 (a)] with (6.2.2) yields a smooth map

$$U_n \colon \lfloor \overline{B_1(0)}, B_{\varepsilon^\#_{1,n}}(0) \rfloor_\infty \to C^\infty(B_1(0), \mathbb{R}^d), U_n(\sigma) := (u_n)_*(\sigma),$$

with $(u_n)_*(\sigma)(x) := u_n(x, \sigma(x))$ for $x \in B_1(0)$. By (6.2.2) U_n maps the zero-map to the zero-map. Evaluating (6.2.2) pointwise for $(X,x) \in H_{R_i^\#} \times \Omega_{1,K_{5,i}}$, the local formula (6.2.2) and Lemma D.0.6 (b) yield the identity $t_n \circ (E^{-1}E^\#)_i = U_n \circ r_n$. Thus $(E^{-1}E^\#)_i$ is a patched mapping, which is smooth on the patches, whence $(E^{-1}E^\#)_i$ is smooth by Proposition C.3.7.

For each $j \in I$ construct in the same manner an open set $H_{R_j^\#} \subseteq \mathfrak{X}\left(\Omega_{5,K_{5,j}}\right)$ together with a smooth map $(E^{-1}E^\#)_j$. Define $H_i^\# := (\mathrm{res}_{\Omega_{5,i}}^{W_{\alpha(i)}})^{-1}(H_{R_i^\#}) \subseteq \mathcal{H}_{R_i^\#} \subseteq \mathfrak{X}\left(W_{\alpha(i)}\right)$ and by Construction 6.1.6 $H^\# := \Lambda_{\mathcal{C}}^{-1}(\oplus_{i\in I} H_i^\#) \subseteq \mathcal{H}^\#$ holds. For each $[\hat{\sigma}] \in H^\#$ the family $((E^{-1}E^\#)_i(\sigma_{\alpha(i)}|_{\Omega_{5,K_{5,i}}})|_{U_i})_{i\in I}$ is a family of vector fields. Since $[\hat{\sigma}]$ is compactly supported, only finitely many canonical lifts $\sigma_{\alpha(i)}$. By standard Riemannian geometry, the Riemannian exponential map composed with the zero section yields the identity. Hence (6.2.3) shows that only finitely many of the vector fields $((E^{-1}E^\#)_i(\sigma_{\alpha(i)}|_{\Omega_{5,K_{5,i}}})|_{U_i})_{i\in I}$ will be non-zero. We claim that these vector fields form a canonical family of an orbisection. If this were true, these vector fields define a compactly supported orbisection $E^{-1}E^\#([\hat{\sigma}])$, whose lifts with respect to $\mathcal{A}$ are given by $((E^{-1}E^\#)_i(\sigma_{\alpha(i)}|_{\Omega_{5,K_{5,i}}})|_{U_i})_{i\in I}$. As $U_i \subseteq \Omega_{1,K_{5,i}}$ holds, these vector fields yield an orbisection, if the following is satisfied:

Let $[\hat{\sigma}] \in H^\#$ *and* $\lambda \in \mathcal{C}h_{W_k,W_l}$ *be a change of charts which satisfies* $\mathrm{dom}\,\lambda \subseteq \Omega_{1,i}$ *and* $\mathrm{cod}\,\lambda \subseteq \Omega_{1,j}$ *for some* $k = \alpha(i)$ *and* $l = \alpha(j)$. *The following identity holds:*

$$T\lambda(E^{-1}E^\#)_i(\sigma_k|_{\Omega_{5,K_{5,i}}})|_{\mathrm{dom}\,\lambda} = (E^{-1}E^\#)_j(\sigma_l|_{\Omega_{5,K_{5,j}}}) \circ \lambda \tag{6.2.4}$$

The argument given in the proof of Lemma 6.1.7 may be repeated almost verbatim: We check the identity (6.2.4) locally: Choose some $x \in \mathrm{dom}\,\lambda \subseteq \Omega_{1,i}$ and a chart $(V_{5,k}^n, \kappa_n^k) \in \mathcal{F}_5(K_{5,i})$ with $x \in V_{1,k}^n$. Again there is some Z_r^k with $V_{5,k}^n \subseteq Z_r^k$. As $x \in V_{1,k}^n \subseteq \mathcal{K}_i^\circ$ and $\lambda(x) \in \mathcal{K}_j$, there is an open embedding of orbifold charts $\mu\colon Z_k^r \to W_l$ with $\mu(x) = \lambda(x)$. After possibly composing μ with a suitable element of H_l, there is an open neighborhood U_x with $\mu|_{U_x} = \lambda|_{U_x}$ and thus $T_x\mu = T_x\lambda$. Since ρ and $\rho^\#$ are Riemannian orbifold metrics, each change of orbifold chart morphism in $\mathcal{C}h_{W_k,W_l}$ is a Riemannian embedding of its domain endowed with the induced metrics into the Riemannian manifold (W_l, ρ_l) respectively $(W_l, \rho_l^\#)$. By construction of $H_i^\#$ each $X \in H_i^\#$ satisfies

$$\left\|\phi_{k,[n],x}^{-1} \exp_{W_k,[n]}^\# X_{[n]}\right\|_{\overline{B_1(0)},0} < \varepsilon_n < R_i \text{ for each } 1 \leq n \leq N_i \tag{6.2.5}$$

Recall from Construction 6.1.1 V. the properties of R_i and S_k:
The definitions imply that $T\mu(E^{-1}E^\#)_i(\sigma_k|_{\Omega_{5,K_{5,i}}})(V_{1,k}^n) \subseteq \hat{O}_l \subseteq \mathrm{dom}\exp_{W_l}$ holds for $[\hat{\sigma}] \in H^\#$. Computing locally on $V_{5,k}^n$, we use that $\mu(\kappa_n^k)^{-1}$ is a Riemannian embedding into W_l. Again by [41, IV. Proposition 2.6] the identity $\exp_{W_j} T\mu(\kappa_n^k)^{-1}(v) = \mu(\kappa_n^k)^{-1}\exp_{k,[n]}(v)$ holds for each $v \in \mathrm{dom}\exp_{W_k,[n]}$. The family $\{\sigma_k\}_{k\in J}$ is a canonical family of lifts, i.e. $\sigma_l\mu = T\mu\sigma_k$ holds. By definition of $H_i^\# \subseteq \mathcal{H}_{R_i^\#}$ the identity $\kappa_n^k \exp_{W_k}^\# \circ X(z) = \exp_{W_k,[n]}^\# T\kappa_n X(z)$ holds for each $z \in V_{3,k}^n$ and $X \in H_i^\#$ (cf. the proof of Lemma 6.1.7). Observe that $\lambda(x) \in \Omega_{1,j}$ and $\sigma_l \in H_j^\#$ hold. Combining these facts one computes:

$$\begin{aligned}
&\exp_{W_l} T_x\lambda(E^{-1}E^\#)_k(\sigma_k|_{\Omega_{5,K_{5,i}}})(x) = \exp_{W_l} T_x\mu(\kappa_n^k)^{-1}\kappa_n^k(\exp_{W_k}|_{N_x})^{-1}\exp_{W_k}^\# \sigma_k(x)\\
=&\mu(\kappa_n^k)^{-1}\exp_{W_k,[n]} T\kappa_n^k(\exp_{W_k}|_{N_x})^{-1}\exp_{W_k}^\#\sigma_k(x) \overset{D.0.6(b)}{=} \mu\exp_{W_k}^\#\sigma_k(x)\\
=&\mu(\kappa_n^k)^{-1}\exp_{W_k,[n]}^\# T\kappa_n^k\sigma_k l(x) = \exp_{W_l}^\#\sigma_l(\mu(x)) = \exp_{W_l}^\#\sigma_l(\lambda(x))\\
\overset{(6.2.3)}{=}&\exp_{W_l}(E^{-1}E^\#)_l(\sigma_l|_{\Omega_{5,K_{5,j}}})(\lambda(x))
\end{aligned}$$

As $x \in \mathcal{K}_k^\circ$ and $\lambda(x) \in \Omega_{1,K_{5,j}}$ hold, the definition of R_i implies $T_x\lambda(E^{-1}E^\#)_k(\sigma_{k|\Omega_{5,K_{5,i}}})(x) \in \hat{O}_l$. By cosntruction of $H_j^\#$, we deduce $(E^{-1}E^\#)_l(\sigma_l|_{\Omega_{5,K_{5,i}}})\lambda(x) \in \hat{O}_l$. As $\exp_l$ is injective on

$T_{\lambda(x)}W_l \cap \hat{O}_l$ and $x \in \operatorname{dom}\lambda$ was arbitrary, this proves (6.2.4). We conclude that the family $((E^{-1}E^{\#})_i(\sigma_{\alpha(i)}|_{\Omega_{5,K_{5,i}}})|_{U_i})_{i\in I}$ is a family of canonical lifts for a compactly supported orbisection $E^{-1}E^{\#}([\hat{\sigma}])$. Define $E^{-1}E^{\#}\colon H^{\#} \to \mathfrak{X}_{\mathrm{Orb}}(Q)_c, [\hat{\sigma}] \mapsto E^{-1}E^{\#}([\hat{\sigma}])$. Using the patchworks $(\lambda_i)_{i\in I}$ and $(\tau_i)_{i\in I}$ for $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$ (see Lemma 6.1.10), a computation yields the identity

$$\operatorname{res}_{U_i}^{\Omega_{1,K_{5,i}}}(E^{-1}E^{\#})_i \operatorname{res}_{\Omega_{5,K_{5,i}}}^{W_{\alpha(i)}} \lambda_i|_{H^{\#}}^{H_i^{\#}} = \tau_i E^{-1}E^{\#}, \quad i \in I.$$

We have already seen that $(E^{-1}E^{\#})_i$ is smooth, such that $(E^{-1}E^{\#})_i(0_{\alpha(i)}) = 0_i$ holds for each $i \in I$. By [25, Lemma F.15 (a)] the mappings $\operatorname{res}_{U_i}^{\Omega_{1,K_{5,i}}}$, $\operatorname{res}_{\Omega_{5,K_{5,i}}}^{W_{\alpha(i)}}$ are smooth, whence $E^{-1}E^{\#}$ is a patched mapping which is smooth on the patches. By Proposition C.3.7 $E^{-1}E^{\#}$ must be smooth and therefore it is continuous. Using continuity, there is an open, connected zero-neighborhood $\mathcal{R}^{\#} \subseteq \mathcal{H}_0^{\#} \cap H^{\#}$ such that $E^{-1}E^{\#}(\mathcal{R}^{\#}) \subseteq E^{-1}(\mathcal{P})$. Uniqueness of canonical lifts proves that the canonical lifts of $E^{-1}E^{\#}([\hat{\sigma}])$ coincide on $\Omega_{1,K_{5,i}}$ with the vector fields $(E^{-1}E^{\#})_i(\sigma_{\alpha(i)}|_{\Omega_{5,K_{5,i}}})$. Recall the construction of the representative $\hat{E}^{\sigma}$ of $E([\hat{\sigma}])$ in Proposition 6.1.4. Using (6.2.3), the construciton yields for $E(E^{-1}E^{\#}([\hat{\sigma}]))$ and $i \in I$ the lifts $\exp^{\#}_{W_{\alpha(i)}} \circ \sigma_i$. The same lifts are obtained, if this construction is carried out with respect to the Riemannian obrifold exponential map $[\exp^{\#}_{\mathrm{Orb}}]$. As orbifold diffeomorphisms are uniquely determined by a family of lifts (cf. Corollary 3.1.11), $E^{\#}([\hat{\sigma}]) = E \circ (E^{-1}E^{\#})([\hat{\sigma}]) \in E(E^{-1}(\mathcal{P})) = \mathcal{P}$ holds for each $[\hat{\sigma}] \in \mathcal{R}^{\#}$. The set $\mathcal{R}^{\#}$ is an open and connected zero-neighborhood contained in $\mathcal{H}_0^{\#}$. Since $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0^{\#}$ is connected, $\langle E^{\#}(\mathcal{R}^{\#})\rangle = \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0^{\#}$ holds by [35, Theorem 7.4], which implies $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0 \subseteq \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$. In particular the inclusion morphism $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0^{\#} \to \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$ is smooth on the open identity-neighborhood $E^{\#}(\mathcal{R}^{\#})$, whence smooth by [9, III. §1, Proposition 4]. Reversing the roles of ρ and $\rho^{\#}$, one deduces that also $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0 \subseteq \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0^{\#}$ and the inclusion morphism $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0 \to \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0^{\#}$ is smooth. In conclusion $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$ and $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0^{\#}$ coincide as Lie groups. □

So far, we achieved that the Lie group structure on $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$ does neither depend on the local data (the atlases $\mathcal{A}$, $\mathcal{B}$ etc.) nor on the Riemannian orbifold metric. We exploit these facts to prove that the requirements of Proposition C.4.3 (b) are satisfied:

6.2.3 Proposition *Let $[\hat{\phi}] \in \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ be an arbitrary orbifold diffeomorphism. Then*

$$c_{[\hat{\phi}]}\colon \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0 \to \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U}), [\hat{f}] \mapsto [\hat{\phi}] \circ [\hat{f}] \circ [\hat{\phi}]^{-1}$$

is a smooth map, whose image is contained in $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$.

Proof. The proof will be quite simple, after some preparations have been dealt with:
Following Corollary 3.1.12 (d), we may choose orbifold atlases $\mathcal{V}_i := \left\{ (V_k^i, L_k^i, \pi_k^i) \in \mathcal{U} \middle| k \in K \right\} \subseteq \mathcal{U}$, $i = 1,2$ together with a representative $\Phi = (\phi, \{\phi_k | k \in K\}, P, \nu) \in \mathrm{Orb}(\mathcal{V}_1, \mathcal{V}_2)$ of $[\hat{\phi}]$ such that each $\phi_k\colon V_k^1 \to V_k^2$ is a diffeomorphism. Furthermore Corollary 3.1.8 assures that we may choose $P = \mathcal{C}h_{\mathcal{V}_1}$ and $\nu(\lambda) = \phi_l \lambda \phi_k^{-1}|_{\phi_k(\operatorname{dom}\lambda)}$ for $\lambda \in \mathcal{C}h_{V_k^1, V_l^1}$.

By Proposition 2.6.7 there are locally finite atlases $\mathcal{A}$ respectively $\mathcal{B}$ indexed by I respectively J which satisfy the properties of the atlases in Construction 6.1.1 I.. In addition there is a map $\beta \colon J \to K$, such that: W_j is an open subset of $V^1_{\beta(j)}$, the inclusion of sets induces an embedding of orbifold charts and $\overline{W_j} \subseteq V^1_{\beta(j)}$ is compact for each $j \in J$. As a consequence of Lemma 6.2.1, we may construct $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$ with respect to these atlases and the Riemannian orbifold metric ρ. Thus there are open sets $\mathcal{H}_1 \subseteq \mathcal{H} := \Lambda_{\mathcal{C}}^{-1}(\mathcal{H}_{R_i})$ and a diffeomorphism $E|_{\mathcal{H}_1}$ onto an identity neighborhood in $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$.
By construction the inclusions of sets $U_i \subseteq W_{\alpha(i)} \subseteq V_{\beta(\alpha(i))}$ and $\phi_{\beta(\alpha(i))}$ are change of orbifold charts for each $i \in I$. For $i \in I$, the sets $W^+_j := \phi_{\beta(\alpha(i))}(W_{\alpha(i)})$ and $U^+_i := \phi_{\beta\alpha(i)}(U_i)$ are $L^2_{\beta(\alpha(i))}$-stable, open and relatively compact subsets of $V^2_{\beta(\alpha(i))}$ (cf. Lemma 3.1.9 (a)). Hence we obtain families of orbifold charts for Q

$$\mathcal{A}^+ := \left\{ (U^+_i, G_i, \pi^2_{\beta\alpha(i)}|_{U^+_i}) \middle| i \in I \right\} \text{ and } \mathcal{B}^+ := \left\{ (W^+_j, H_j, \varphi^+_j := \pi^2_{\beta(j)}|_{W^+_j}) \middle| j \in J \right\}.$$

The underlying map ϕ is a homeomorphism and each ϕ_k is a diffeomorphism. Hence $\mathcal{A}^+$ and $\mathcal{B}^+$ are orbifold atlases for Q, such that $\overline{U^+_i} \subseteq W^+_{\alpha(i)}$ holds for each $i \in I$ and the inclusions of sets induce embeddings of orbifold charts. Since W^+_j is a relatively compact subset of $V^2_{\beta(j)}$ for each $j \in J$, we deduce from the continuity of $\pi^2_{\beta(j)}$ and [20, Corollary 3.1.11] that the image of each chart in $\mathcal{A}^+$ and $\mathcal{B}^+$ is relatively compact. Exploiting that ϕ is a homeomorphism, $\mathcal{A}^+$ and $\mathcal{B}^+$ are locally finite atlases, since the same holds for $\mathcal{A}$ and $\mathcal{B}$. Furthermore by construction of $\mathcal{A}$ and $\mathcal{B}$, for each connected component $C \subseteq Q$, there is a point z_C which is only contained in the images of a unique pair of charts in $\mathcal{A} \times \mathcal{B}$. The homeomorphism ϕ permutes the connected components of Q, whence each z_C is mapped into a separate component. Each element of $\{ \phi(z_C) | C \subseteq Q \text{ connected component} \}$ is thus contained in the images of a unique pair in $\mathcal{A}^+ \times \mathcal{B}^+$, such that the images of different pairs are contained in different connected components. Summing up, the atlases $\mathcal{A}^+$ and $\mathcal{B}^+$ satisfy all properties required in 6.1.1 I..

As $\mathcal{B}$ is an atlas, a family of lifts for a representative of $[\hat{\phi}]$ is given by $\left\{ \Phi_j := \phi_{\beta(j)}|_{W_j} \middle| j \in J \right\}$. By construction each of these lifts is a diffeomorphism and $\Phi_{\alpha(i)}(U_i) = U^+_i$ holds for each $i \in I$. Corollary 3.1.12 assures that $\left\{ \Phi_j^{-1} \middle| j \in J \right\}$ is a family of lifts for a representative of $[\hat{\phi}]^{-1}$ in $\mathrm{Orb}(\mathcal{B}^+, \mathcal{B})$. Observe that $\Phi_j^{-1}(U^+_i) = U_i$ holds for each $i \in \alpha^{-1}(j)$. Before we prove the smoothness of $c_{[\hat{\phi}]}$, consider the following auxilliary maps:
Define $t_i \colon \mathcal{H}_{R_i} \to \mathfrak{X}\left(U^+_i\right), X \mapsto T\Phi_{\alpha(i)} X \Phi^{-1}_{\alpha(i)}|_{U^+_i}$ for $i \in I$. For $[\hat{\sigma}] \in \mathcal{H}$, the family $\{ t_i(\sigma_i) | i \in I \}$ defines a family of vector fields. We claim that these vector fields are a family of canonical lifts of an orbisection: Let $\lambda \in \mathcal{C}h_{U^+_i, U^+_j}$ be any change of charts with $i, j \in I$ arbitrary. As noted above, $\mu := \Phi^{-1}_{\alpha(j)} \lambda \Phi_{\alpha(i)}|_{\Phi^{-1}_{\alpha(i)}(\mathrm{dom}\,\lambda)}$ is a change of charts in $\mathcal{C}h_{U_i,U_j}$ and we compute

$$\begin{aligned} t_j(\sigma_j) \circ \lambda &= T\Phi_{\alpha(j)} \sigma_j \Phi_{\alpha(j)}|_{U^+_j} \lambda = T\Phi_{\alpha(j)} \sigma_j \mu \Phi^{-1}_{\alpha(i)}|_{\mathrm{dom}\,\lambda} \\ &= T\Phi_{\alpha(j)} T\mu \sigma_i \Phi^{-1}_{\alpha(i)}|_{\mathrm{dom}\,\lambda} = T\lambda t_i(\sigma_i)|_{\mathrm{dom}\,\lambda}. \end{aligned}$$

The family $\{ t_i(\sigma_i) | i \in I \}$ is a family of canonical lifts with respect to $\mathcal{A}^+$, whence it induces a unique orbisection $t([\hat{\sigma}])$. By construction $t_i(\sigma_i)$ will be the zero-section if σ_i is the zero-section.

Hence $t([\hat{\sigma}])$ is compactly supported and we obtain a map $t\colon \mathcal{H} \to \mathfrak{X}_{\mathrm{Orb}}(Q)_c, [\hat{\sigma}] \mapsto t([\hat{\sigma}])$. Consider the patchwork induced by the maps

$$p_i\colon \mathfrak{X}_{\mathrm{Orb}}(Q)_c \to \mathfrak{X}\left(W_{\alpha(i)}\right), p_i([\hat{\sigma}]) = \sigma_{\alpha(i)} \text{ and } q_i\colon \mathfrak{X}_{\mathrm{Orb}}(Q)_c \to \mathfrak{X}\left(U_i^+\right), q_i([\hat{\sigma}]) = \sigma_{U_i^+},\ i \in I,$$

sending an orbisection to their canonical lifts. By construction of $\mathcal{H}$ (cf. Construction 6.1.6), $p_i(\mathcal{H}) \subseteq \mathcal{H}_{R_i}$ holds. From $t_i \circ p_i|_{\mathcal{H}}^{\mathcal{H}_{R_i}} = q_i \circ t$ we deduce that t is a patched mapping. We claim that t_i is smooth for each $i \in I$. If this were true, this implies the smoothness of t by Proposition C.3.7. To prove the claim, consider $t_i'\colon \mathcal{H}_{R_i}^{\Omega_{5,K_{5,i}}} \to \mathfrak{X}\left(\Phi_{\alpha(i)}(\Omega_{5,K_{5,i}})\right), X \mapsto T\Phi_{\alpha(i)}X\Phi_{\alpha(i)}^{-1}|_{\Phi_{\alpha(i)}(\Omega_{5,K_{5,i}})}$ and note the identity $t_i = \mathrm{res}_{U_i^+}^{\Phi_{\alpha(i)}(\Omega_{5,K_{5,i}})} t_i' \,\mathrm{res}_{\Omega_{5,K_{5,i}}}^{W_{\alpha(i)}}$. Since the restriction maps are smooth, it suffices to prove the smoothness of t_i'. By construction $\Omega_{5,K_{5,i}}$ is covered by the finite family of manifold charts $\mathcal{F}_5(K_{5,i}) = \left\{ (V_{5,\alpha(i)}^n, \kappa_n^{\alpha(i)}) \middle| 1 \leq n \leq N_i \right\}$. Hence the sets $V_{5,\alpha(i)}^{n,+} := \Phi_{\alpha(i)}(V_{5,\alpha(i)}^n)$ cover $\Phi_{\alpha(i)}(\Omega_{5,K_{5,i}})$. Set $\gamma_n^{\alpha(i)} := \kappa_n^{\alpha(i)}\Phi_{\alpha(i)}^{-1}|_{V_{5,\alpha(i)}^{n,+}}$ to obtain a manifold atlas for $\Phi_{\alpha(i)}(\Omega_{5,K_{5,i}})$: $\mathcal{F}_5^+(K_{5,i}) := \left\{ (V_{5,\alpha(i)}^{n,+}, \gamma_n^{\alpha(i)}) \middle| 1 \leq n \leq N_i \right\}$. By Definition C.3.1 there are finite families of linear continuous mappings $\theta_{\alpha(i)}^n\colon \mathfrak{X}\left(\Omega_{5,K_{5,i}}\right) \to C^\infty(V_{5,\alpha(i)}^n, \mathbb{R}^d), X \mapsto X_{\kappa_n}$ and $\theta_{\alpha(i)}^{n,+}\colon \mathfrak{X}\left(\Phi_{\alpha(i)}\Omega_{5,K_{5,i}}\right) \to C^\infty(V_{5,\alpha(i)}^{n,+}, \mathbb{R}^d), Y \mapsto Y_{\gamma_n}$, with $1 \leq n \leq N_i$. The family $(\theta_{\alpha(i)}^n)_{1 \leq n \leq N_i}$ is a patchwork for $\mathfrak{X}\left(\Omega_{5,K_{5,i}}\right)$ and $(\theta_{\alpha(i)}^{n,+})_{1 \leq n \leq N_i}$ is a patchwork for $\mathfrak{X}\left(\Phi_{\alpha(i)}(\Omega_{5,K_{5,i}})\right)$ by Lemma [25, Lemma F.6]. As $\Phi_{\alpha(i)}^{-1}$ is smooth, the pullback $C^\infty(\Phi_{\alpha(i)}^{-1}|_{V_{5,\alpha(i)}^{n,+}}^{V_{5,\alpha(i)}^n}, \mathbb{R}^d)$ is continuous linear and therefore smooth by [23, Lemma 3.7]. A quick computation yields for $1 \leq n \leq N_i$ the identity $\theta_{\alpha(i)}^{n,+} \circ t_i' = C^\infty(\Phi_{\alpha(i)}^{-1}|_{V_{5,\alpha(i)}^{n,+}}^{V_{5,\alpha(i)}^n}, \mathbb{R}^d) \circ \theta_{\alpha(i)}^n$. We conclude that t_i is a patched mapping, which is smooth on the patches, whence smooth by Proposition C.3.7.

The orbifold diffeomorphism $[\hat{\phi}]^{-1}$ induces a unique pullback metric $\rho^\# := ([\hat{\phi}]^{-1})^*\rho$ on Q (cf. Lemma 5.0.8). Denote by ρ_j the members of ρ on the orbifold charts $(W_j, H_j, \varphi_j),\ j \in J$. The Riemannian metricw associated to $\rho^\#$ with respect to $(W_j^+, H_j, \varphi_j^+),\ j \in J$ are given by the pullback metric $\rho_j^\# := (\Phi_j^{-1})^*\rho_j$. For $j \in J$ let $\exp_j\colon D_j \to W_j$ be the Riemannian exponential maps with respect to (W_j, ρ_j) and $\exp_j^\#\colon D_j^\# \to W_j^+$ be the exponential map with respect to $(W_j^+, \rho_j^\#)$. These pullback metrics turn Φ_j, Φ_j^{-1} into Riemannian isometries and the map $[\hat{\phi}]$ into an orbifold isometry. In particular we derive $T\Phi_j(D_j) = D_j^\#$ and the exponential identity

$$\exp_j^\#(T\Phi_j)|_{D_j}^{D_j^\#} = \phi_j \exp_j .$$

Let $[\hat{\sigma}]$ be in $\mathcal{H}$ and consider e^{σ_i} as in Proposition 6.1.4. From the last identity we deduce

$$\Phi_{\alpha(i)} \circ e^{\sigma_i} \circ \Phi_{\alpha(i)}|_{U_i^+} = \Phi_{\alpha(i)} \exp_{\alpha(i)} \sigma_i \Phi_{\alpha(i)}^{-1}|_{U_i^+} = \exp_{\alpha(i)}^\# T\Phi_{\alpha(i)}\sigma_i\Phi_{\alpha(i)}^{-1}|_{U_i^+}. \tag{6.2.6}$$

Combining Lemma 6.2.1 with Lemma 6.1.2, one may construct $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$ with respect to the atlases $\mathcal{A}^+$, $\mathcal{B}^+$ and the Riemannian orbifold metric $\rho^\#$. Hence there are an open connected zero-neighborhood $H_\#^+ \subseteq \mathfrak{X}_{\mathrm{Orb}}(Q)_c$ and a map $E_\#^+\colon H_\#^+ \to \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0, [\hat{\sigma}] \mapsto [\exp_{\mathrm{Orb}}^\#] \circ [\hat{\sigma}]|^{\Omega^\#}$.

Here $[\exp^{\#}_{\mathrm{Orb}}]$ is the Riemannian orbifold exponential map associated to $\rho^{\#}$, whose domain is $\Omega^{\#}$. The map $E^{+}_{\#}$ is a diffeomorphism onto its image, which is an open identity-neighborhood in $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$. As t is smooth and thus continuous, there is an open connected zero-neighborhood $A \subseteq \mathcal{H}_1$, such that $t(A) \subseteq H^{+}_{\#}$ holds.
Recall from Corollary 3.1.11 that an orbifold diffeomorphism is uniquely determined by the family of lifts of any of its representatives. Hence for $[\hat{\sigma}] \in \mathcal{H}_1 = E^{-1}(\mathcal{P})$ (cf. Proposition 6.1.11), the orbifold diffeomorphism $[\hat{\phi}] \circ E([\hat{\sigma}]) \circ [\hat{\phi}]^{-1}$ is uniquely determined by $\left\{ \Phi_{\alpha(i)} \circ e^{\sigma_i} \circ \Phi_{\alpha(i)}|_{U_i^+} \middle| i \in I \right\}$. In Proposition 6.1.4 a representative of $E^{+}_{\#}([\hat{\sigma}])$, $[\hat{\sigma}] \in H^{+}_{\#}$ in $\mathrm{Orb}(\mathcal{A}^+, \mathcal{B}^+)$ has been explicitely computed. Its lifts were given by the family $\left\{ \exp^{\#}_{\alpha(i)} \circ \sigma_{U_i^+} \right\}_{i \in I}$. Since the lifts uniquely determine the diffeomorphism, equation (6.2.6) implies $c_{[\hat{\phi}]}E([\hat{\sigma}]) = E^{+}_{\#}t([\hat{\sigma}]) \in \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$ for every $[\hat{\sigma}] \in A$. In particular $c_{[\hat{\phi}]}E(A) \subseteq \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$ holds. The set $E(A)$ is an open connected identity-neighborhood, whence it generates the connected Lie group $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$ by [35, Theorem 7.4]. Therefore $c_{[\hat{\phi}]}(\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0) = c_{[\hat{\phi}]}(\langle E(A) \rangle) \subseteq \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$ holds.<We deduce from $c_{[\hat{\phi}]}|_{E(A)} = E^{+}_{\#} \circ t|_A^{H^{+}_{\#}} \circ (E|_A^{E(A)})^{-1}$ that the group automorphism $c_{[\hat{\phi}]}$ of $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$ is smooth on the open identity neighborhood $E(A)$, whence smooth by [9, III. §1, Proposition 4]. □

The preceding Proposition shows that for each $[\hat{\phi}]$, we may choose an open identity-neighborhood $W_{\hat{\phi}} := c^{-1}_{[\hat{\phi}]}(U)$, such that $c_{[\hat{\phi}]}|_{W_{\hat{\phi}}} \colon W_{\hat{\phi}} \to U$ is smooth. All requirements of Proposition C.4.3 (b) have been checked. Applying this construction principle, we obtain a unique Lie group structure on $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$, turning $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$ into an open submanifold of $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$. Summarizing the results, we obtain the following

6.2.4 Theorem *The group* $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ *can be made into a Lie group in a unique way, such that the following condition is satisfied:*
For some Riemannian orbifold metric ρ *on* $(Q,\mathcal{U})$*, let* $[\exp_{\mathrm{Orb}}]$ *be the Riemannian orbifold exponential map with domain* Ω*. There exists an open zero-neighborhood* $\mathcal{H}_\rho$ *in* $\mathfrak{X}_{Orb}(Q)_c$ *such that* $[\hat{\sigma}] \mapsto [\exp_{\mathrm{Orb}}] \circ [\hat{\sigma}]|^{\Omega}$ *is a well defined* C^∞*-diffeomorphism of* $\mathcal{H}_\rho$ *onto an open submanifold of* $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$*.*
The condition is then satisfied for every Riemannian orbifold metric on $(Q,\mathcal{U})$*. The identity component of* $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ *is the Lie group* $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$ *constructed in Section 6.1.*

6.2.5 Corollary *If* $(Q,\mathcal{U})$ *is a compact orbifold, the Lie group* $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ *is a Fréchet Lie group.*

Proof. If Q is compact, by Corollary 4.3.6 $\mathfrak{X}_{\mathrm{Orb}}(Q)_c = \mathfrak{X}_{\mathrm{Orb}}(Q)$ is a Fréchet space. □

We shall now consider subgroups of $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$, which will turn out to be Lie subgroups of the Lie group $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$.

6.2.6 Definition Let $K \subseteq Q$ be a compact subset and denote for an orbifold map $[\hat{f}]$ its underlying map by f. Define the *set of all orbifold diffeomorphisms whose support is contained in* K:

$$\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_K := \left\{ [\hat{f}] \in \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U}) \middle| f|_{Q\setminus K} \equiv \mathrm{id}_Q \right\}$$

We also say that the elements of $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_K$ coincide with the identity morphism of Q off K. Furthermore we define the subset $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_c \subseteq \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ of all orbifold diffeomorphisms, whose underlying map coincides with id_Q outside some compact set in Q. Observe that the sets $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_K, \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_c$ are subgroups of $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$.

6.2.7 Remark Notice that by construction $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_c$ contains $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$. The subgroup $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_c$ therefore is an open subgroup of $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ by [35, Theorem 5.5.]. Hence it becomes a Lie group when considered as an open submanifold.

6.2.8 Proposition *Each compact subset K of Q is contained in a compact set compact set L, such that the group $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_L$ is a closed Lie subgroup of $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ modelled on $\mathfrak{X}_{Orb}(Q)_L$.*

Proof. We shall again use the notation of Section 6.1. The atlas $\mathcal{A}$ is locally finite and the image of each chart in $\mathcal{A}$ is relatively compact. Thus there are only finitely many charts (U_i, G_i, ψ_i) in $\mathcal{A}$ with $\psi_i(U_i) \cap K \neq \emptyset$. Let I_K be the set indexing this family and consider the closed set $L := Q \setminus \left(\bigcup_{i \in I \setminus I_K} \psi_i(U_i)\right)$. By construction $K \subseteq L \subseteq \overline{\bigcup_{i \in I_K} \psi_i(U_i)}$ holds, whence L is a compact set. We claim that $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_L$ is a closed Lie subgroup of $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ modelled on $\mathfrak{X}_{\mathrm{Orb}}(Q)_L$. Choose for each $i \in I \setminus I_K$ a non singular point $x_i \in U_i$. By [39, Theorem 1.9.5] we may choose $\varepsilon_i > 0$ with $\exp_{W_{\alpha(i)}}(B_{\rho_{\alpha(i)}}(0_{x_i}, \varepsilon_i)) \cap H_{\alpha(i)}.x_i = \{ x_i \}$. By definition of the topology on $\mathfrak{X}\left(W_{\alpha(i)}\right)$, there is an open neighborhood $\mathcal{R}_i \subseteq \mathfrak{X}\left(W_{\alpha(i)}\right)$ of the zero-section, such that $\sigma \in \mathcal{R}_i$ implies $\sigma(x_i) \in B_{\rho_{\alpha(i)}}(0_{x_i}, \varepsilon_i)$. Define the open neighborhood of the zero-orbisection

$$\mathcal{R} := \Lambda_{\mathcal{C}}^{-1}\left(\bigoplus_{i \in I \setminus I_K} \mathcal{R}_i \oplus \bigoplus_{j \in I_K} \mathfrak{X}\left(W_{\alpha(j)}\right)\right) \subseteq \mathfrak{X}_{\mathrm{Orb}}(Q)_c.$$

Let $[\hat{\sigma}]$ be an element of $\mathcal{H}_1 \cap \mathcal{R}$, where $\mathcal{H}_1$ is the open zero-neighborhood defined in Proposition 6.1.11. Denote by $\{ \sigma_i | i \in I \}$ the family of canonical lifts of $[\hat{\sigma}]$ with respect to $\mathcal{A}$. Recall that $E([\hat{\sigma}])$ is a diffeomorphism, whose local lift with respect to $(U_i, G_i, \psi_i), i \in I \setminus I_K$ is the map $e^{\sigma_i} = \exp_{W_{\alpha(i)}}|_{\hat{O}_{\alpha(i)}} \circ \sigma_i$. Furthermore $\exp_{W_{\alpha(i)}}|_{\hat{O}_{\alpha(i)} \cap T_x W_{\alpha(i)}}$ is a diffeomorphism for each $x \in U_i$ mapping 0_x to x. Since the canonical lift with respect to (U_i, G_i, ψ_i) of the zero-orbisection is the zero-section, we deduce that $E(\mathcal{H}_1 \cap \mathcal{R} \cap \mathfrak{X}_{\mathrm{Orb}}(Q)_L) \subseteq \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_L$ holds.
On the other hand consider $[\hat{\sigma}] \in \mathcal{H}_1 \cap \mathcal{R}$ with $E([\hat{\sigma}]) \in \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_L$. The underlying map of $E([\hat{\sigma}])$ coincides with id_Q on $Q \setminus L$. By construction $\psi_i(U_i) \subseteq L$ holds for each $i \in I \setminus I_K$. Hence $\varphi_{\alpha(i)} \circ e^{\sigma_i} = \mathrm{id}_Q \circ \psi_i = \psi_i$ holds. We deduce that $e^{\sigma_i} \colon U_i \to W_{\alpha(i)}$ must be an embedding of orbifold charts. Since the canonical inclusion $U_i \to W_{\alpha(i)}$ is an embedding of orbifold charts by Construction 6.1.1 I.(d), Proposition 2.2.2 (d) yields $e^{\sigma_i} = h|_{U_i}$ for some $h \in H_{\alpha(i)}$. Specializing to $x_i \in U_i$ this yields $e^{\sigma_i}(x_i) = h(x_i) \in H_{\alpha(i)}.x_i$. Since $[\hat{\sigma}]$ is contained in $\mathcal{R}$, $\sigma_i \in \mathcal{R}_i$ and thus

$e^{\sigma_i}(x_i) \cap H_{\alpha(i)}.x_i = \{ x_i \}$ holds. We obtain $h(x_i) = x_i$ and since x_i is non singular, $h = \mathrm{id}_{W_{\alpha(i)}}$ follows. Thus $e^{\sigma_i} = \mathrm{id}_{W_{\alpha(i)}}|_{U_i}$ and we deduce that σ_i must be the zero-section in $\mathfrak{X}(U_i)$. Repeate the argument for each $i \in I \setminus I_K$. As $Q \setminus L = \bigcup_{i \in I \setminus I_K} \psi_i(U_i)$ holds by construction, $[\hat{\sigma}]$ is an element of $\mathfrak{X}_{\mathrm{Orb}}(Q)_L$. Summarizing the results so far we obtain:

$$E(\mathcal{H}_1 \cap \mathcal{R}) \cap \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_L = E(\mathcal{H}_1 \cap \mathcal{R} \cap \mathfrak{X}_{\mathrm{Orb}}(Q)_L) \tag{6.2.7}$$

Since the $\mathcal{P} = E(\mathcal{H}_1)$ generates $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_0$, we deduce that $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_L$ is a Lie subgroup of $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$, modelled on $\mathfrak{X}_{\mathrm{Orb}}(Q)_L$. The space $\mathfrak{X}_{\mathrm{Orb}}(Q)_L$ is a closed vector subspace of $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$ by Lemma 4.3.7. Hence the identity (6.2.7) implies that $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_L$ is locally closed in the topological group $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ and thus $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})_L$ is a closed subgroup by [8, III. §2, No. 1 Proposition 4]. □

For a trivial orbifold (i.e. a manifold) one need not refine the zero-neighborhood in the above Proposition. Hence we may always choose $K = L$ in Proposition 6.2.8 for a trivial orbifold.

6.2.9 Remark As mentioned in the introduction, this is not the first work, which considers Lie group structures on the diffeomorphism group of an orbifold. In [6] and the follow up [7], the diffeomorphism group of a compact orbifold has been turned into a convenient Fréchet Lie group. We must mention that the article [6] contains several errors, making it unclear, whether the methods outlined in [6, 7] turn the orbifold diffeomorphism group into a convenient Lie group. To illustrate our concerns, we point out two serious problems in the exposition of [6]:

- Lemma 23 in [6] states that the local lifts of an orbifold map are independent of local chart once chosen. The assertion clarifies the definition of of an orbifold maps as in [6]. However, the assertion of the Lemma is false. A counter example may be obtained as follows: Let $\mathbb{R}^2/G$ be the orbifold induced by the action of the finite group G, generated by a reflections and a rotation of order 4. Consider a smooth map $f\colon]-1,3[\to \mathbb{R}^2$ with $\mathrm{Im}\, f \subseteq \mathbb{R} \times \{ 0 \}$ and $\|f(t)\| > 0$ if and only if $t \in]0,1[\cup]2,3[$. If $q\colon \mathbb{R}^2 \to \mathbb{R}^2/G$ is the global chart for this orbifold, $q \circ f$ is a continuous map, which induces a morphism of orbifolds in the sense of [6]. In fact we may choose for example $f|_{]-0.5,1.5[}$ as a smooth lift at 0. One deduces that there are several possibilities to extend this lift to the pair of charts $]-1,3[, \mathbb{R}^2$ thus contradicting the Lemma.
- In Definition 31 of [6] the space of C^r-orbifold morphisms $C^r_{\mathrm{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$ is endowed with a topology. The topology is defined via the construction of a neighborhood base which depends on a fixed locally finite covering $\mathcal{C}$ of the orbifold $\mathcal{O}_1$. Since the covering $\mathcal{C}$ is fixed, the sets defined in Definition 31 will in general **not** contain all elements of $C^r_{\mathrm{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$. To see this consider the manifold case, explicitely the space $C^r(\mathbb{R}, \mathbb{S}^1)$. Here $\mathbb{S}^1$ is the circle with the usual manifold structure turning it into a one-dimensional smooth manifold. Cover $\mathbb{R}$ by some locally finite covering with compact sets I_n and choose a C^∞-map $f \in C^r(\mathbb{R}, \mathbb{S}^1)$, such that $f(I_n) = \mathbb{S}^1$ holds for some I_n. Since $\mathbb{S}^1$ is not covered by a single manifold chart of $\mathbb{S}^1$, Definition 31 in [6] implies that f is not contained in any basic set defined there (not even in basic neighborhoods around itself!). Hence Definition 31 will in general not yield a "neighborhood base" (or a topology) on $C^r_{\mathrm{Orb}}(\mathcal{O}_1, \mathcal{O}_2)$.
  Unfortunately this "topology" is used in [6] and [7] to obtain a topology on the diffeomorphism group of a compact orbifold, which turns this group into a convenient Lie group.

6.3. The Lie-Algebra of $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$

In this section the Lie-algebra $L(G)$ of the group $G := \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ constructed in Section 6.2 will be characterized. We stick to the notation introduced in Sections 6.1 and 6.2. By construction the map $E\colon \mathfrak{X}_{\mathrm{Orb}}(Q)_c \supseteq \mathcal{H}_1 \to \mathcal{P} \subseteq G, [\hat{\sigma}] \mapsto [\exp_{\mathrm{Orb}}] \circ [\hat{\sigma}]|^{\Omega}$ is a diffeomorphism of the open zero-neighborhood $\mathcal{H}_1$ to an open identity-neighborhood $\mathcal{P}$ in G. Furthermore E maps $\mathbf{0}_{\mathrm{Orb}}$ to $\mathrm{id}_{(Q,\mathcal{U})}$ by Proposition 6.1.5. Use the natural isomorphism $T_{\mathbf{0}_{\mathrm{Orb}}}E$ to identify $T_{\mathrm{id}_{(Q,\mathcal{U})}}G$ with $\mathfrak{X}_{\mathrm{Orb}}(Q)_c \cong T_{\mathbf{0}_{\mathrm{Orb}}}\mathfrak{X}_{\mathrm{Orb}}(Q)_c$.

We modify the classical argument to compute the Lie algebra of the diffeomorphism group of a compact manifold via the adjoint action by Milnor (see [46, pp.1035-1036]). To compute the Lie bracket, we have to understand the adjoint action of $T_{\mathrm{id}_{(Q,\mathcal{U})}}G$ on itself. Using the chart E, the product on G lifts back to a smooth product operation

$$[\hat{\sigma}] * [\hat{\tau}] := E^{-1}(E([\hat{\sigma}]) \circ E([\hat{\tau}]))$$

on the open zero-neighborhood $\{\, ([\hat{\sigma}],[\hat{\tau}]) | E([\hat{\sigma}]) \circ E([\hat{\tau}]) \in \operatorname{Im} E \,\} \subseteq \mathfrak{X}_{\mathrm{Orb}}(Q)_c \times \mathfrak{X}_{\mathrm{Orb}}(Q)_c$. By construction $[\hat{\sigma}] * \mathbf{0}_{\mathrm{Orb}} = [\hat{\sigma}] = \mathbf{0}_{\mathrm{Orb}} * [\hat{\sigma}]$ holds. Hence the constant term of the taylor series of $*$ in $\mathbf{0}_{\mathrm{Orb}}$ (cf. [22, Proposition 1.17]) vanishes. Following [50, Example II.1.8] the taylor series is given as

$$[\hat{\sigma}] * [\hat{\tau}] = ([\hat{\sigma}] + [\hat{\tau}]) + b([\hat{\sigma}],[\hat{\tau}]) + \cdots .$$

Here $b([\hat{\sigma}],[\hat{\tau}]) = \frac{\partial^2}{\partial s \partial t}\Big|_{t,s=0} (t[\hat{\sigma}] * s[\hat{\tau}])$ is a continuous $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$-valued bilinear map and the dots stand for terms of higher degree (cf. [30]). Identify $T_{\mathrm{id}_{(Q,\mathcal{U})}}G$ via $T_{\mathbf{0}_{\mathrm{Orb}}}E$ with $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$. With arguments as in [46, p. 1036], the adjoint action of $T_{\mathrm{id}_{(Q,\mathcal{U})}}G$ on itself is given by

$$\mathrm{ad}([\hat{\sigma}])[\hat{\tau}] = b([\hat{\sigma}],[\hat{\tau}]) - b([\hat{\tau}],[\hat{\sigma}]).$$

In other words, the the skew-symmetric part of the bilinear map b defines the adjoint action.

By [46, Assertion 5.5] (or [50, Example II.3.9]) the Lie Algebra $L(G)$ of G may be identified with $T_{\mathrm{id}_{(Q,\mathcal{U})}}G$, such that the Lie bracket coincides with the adjoint action: $[x,y] = \mathrm{ad}(x)y$. To compute the Lie bracket $[\cdot,\cdot]$ it is sufficient to compute the second derivative of the local product operation in $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$. Consider the atlas $\mathcal{A}$ as in Construction 6.1.1 together with the linear topological embedding with closed image $\Lambda_{\mathcal{A}}\colon \mathfrak{X}_{\mathrm{Orb}}(Q)_c \to \bigoplus_{i\in I} \mathfrak{X}(U_i), [\hat{\sigma}] \mapsto (\sigma_i)_{i\in I}$. For fixed $[\hat{\sigma}],[\hat{\tau}] \in \mathfrak{X}_{\mathrm{Orb}}(Q)_c$, the map $(t,s) \mapsto t[\hat{\sigma}] * s[\hat{\tau}]$ factors through a finite subproduct of the direct sum. Hence the derivative of $s[\hat{\sigma}]*t[\hat{\tau}]$, may thus be computed as the derivatives of the canonical lifts $(t[\hat{\sigma}]*s[\hat{\tau}])_i$. Recall from Lemma 6.1.9 that for each pair $[\hat{\sigma}],[\hat{\tau}] \in \mathcal{H}_1$ there is an orbisection $[\widehat{\sigma \diamond \tau}] \in \mathfrak{X}_{\mathrm{Orb}}(Q)_c$ such that $E([\widehat{\sigma \diamond \tau}]) = E([\hat{\sigma}]) \circ E([\hat{\tau}])$ holds. The mapping E is bijective, whence we deduce for $i \in I$ the identity

$$(t[\hat{\sigma}] * s[\hat{\tau}])_i = (t\sigma \diamond s\tau)_i = (t\sigma_{\alpha(i)} \diamond_i s\tau_{\alpha(i)})|_{U_i}.$$

For the rest of the proof, fix $i \in I$ and compute $\frac{\partial^2}{\partial s \partial t}\Big|_{t,s=0} (t\sigma_{\alpha(i)} \diamond_i s\tau_{\alpha(i)})|_{U_i}$. By construction the vector field $t\sigma_{\alpha(i)} \diamond_i s\tau_{\alpha(i)}$ is defined on $\Omega_{\frac{5}{4},K_{5,i}}$. As the restriction map $\mathrm{res}_{U_i}^{\Omega_{\frac{5}{4},K_{5,i}}}$ is continuous linear by [25, Lemma F.15 (a)], it commutes with the differential, i.e.

$$\mathrm{res}_{U_i}^{\Omega_{\frac{5}{4},K_{5,i}}} \frac{\partial^2}{\partial s \partial t}\Big|_{t,s=0} t\sigma_{\alpha(i)} \diamond_i s\tau_{\alpha(i)} = \frac{\partial^2}{\partial s \partial t}\Big|_{t,s=0} (t\sigma_{\alpha(i)} \diamond_i s\tau_{\alpha(i)})|_{U_i}$$

holds. Thus it suffices to compute its derivative in $\mathfrak{X}\left(\Omega_{\frac{5}{4},K_{5,i}}\right)$.
The family $\left\{ (V^n_{\frac{5}{4},\alpha(i)}, \kappa_n^{\alpha(i)}) \middle| (V^n_{5,\alpha(i)}, \kappa_n^{\alpha(i)})) \in \mathcal{F}_5(K_{5,i} \right\}$ is finite and covers $\Omega_{\frac{5}{4},K_{5,i}}$. Hence the topology on the space $\mathfrak{X}\left(\Omega_{\frac{5}{4},K_{5,i}}\right)$ is induced by the linear embedding with closed image

$$\Gamma\colon \mathfrak{X}\left(\Omega_{\frac{5}{4},K_{5,i}}\right) \to \prod_{(V^n_{5,\alpha(i)},\kappa_n^{\alpha(i)})\in\mathcal{F}_5(K_{5,i})} C^\infty(V^n_{\frac{5}{4},\alpha(i)}, \mathbb{R}^d), X \mapsto (\mathrm{pr}_2 T\kappa_n^{\alpha(i)} X|_{V^n_{\frac{5}{4},\alpha(i)}})_{\mathcal{F}_5,K_{5,i}}.$$

Here pr_2 is the linear projection onto the second component of $B_{\frac{5}{4}}(0) \times \mathbb{R}^d$. Since $\kappa_n^{\alpha(i)}|_{B_{\frac{5}{4}}(0)}$ is a diffeomorphism onto $V^n_{\frac{5}{4},\alpha(i)}$, the mapping

$$C^\infty((\kappa_n^{\alpha(i)})^{-1}|_{B_{\frac{5}{4}}(0)}, \mathbb{R}^d)\colon C^\infty(V^n_{\frac{5}{4},\alpha(i)}, \mathbb{R}^d) \to C^\infty(B_{\frac{5}{4}}(0), \mathbb{R}^d), X \mapsto X \circ (\kappa_n^{\alpha(i)})^{-1}|_{B_{\frac{5}{4}}(0)}$$

is an isomorphism of topological vector spaces by [25, Lemma A.1]. We derive an embedding of topological vector spaces with closed image $(C^\infty(\kappa_n^{\alpha(i)})^{-1}|_{B_{\frac{5}{4}}(0)}, \mathbb{R}^d))_{\mathcal{F}_5(K_{5,i}} \circ \Gamma$. Using this map, the derivative may be computed locally in $A := \prod_{\mathcal{F}_5(K_{5,i})} C^\infty(B_{\frac{5}{4}}(0), \mathbb{R}^d)$. For $X \in \mathfrak{X}\left(W_{\alpha(i)}\right)$ define $X_{[n]} := \mathrm{pr}_2 T\kappa_n^{\alpha(i)} X(\kappa_n^{\alpha(i)})^{-1}|_{B_{\frac{5}{4}}(0)} \in C^\infty(B_{\frac{5}{4}}(0), \mathbb{R}^d)$. The map pr_2 is linear and each $T\kappa_n^{\alpha(i)}$ is linear in the vector space component. Hence the definition of the vector space operations of $\mathfrak{X}\left(W_{\alpha(i)}\right)$ shows that the identity $(tX)_{[n]} = tX_{[n]}$ holds for each $t \in \mathbb{R}$ and $X \in \mathfrak{X}\left(W_{\alpha(i)}\right)$.
To compute the derivative of $(t\sigma_{\alpha(i)} \diamond_i s\tau_{\alpha(i)})$ in A, more information on $(t\sigma_{\alpha(i)} \diamond_i s\tau_{\alpha(i)})_{[n]}$ is needed. Fortunately by Construction 6.1.6 a local formula is available. To write it down explicitely, we need to recall notation and facts from the construction:
For each chart $(V^n_{5,\alpha(i)}, \kappa_n^{\alpha(i)})$ let $\exp_n$ be the Riemannian exponential map on $B_5(0)$ associated to the pullback metric with respect to $\kappa_n^{\alpha(i)}$ and the member of the orbifold metric $\rho_{\alpha(i)}$ on $W_{\alpha(i)}$. Recall from Construction D.0.8 that for $x \in V^n_{\frac{5}{4},\alpha(i)}$ there is an open set $N_x \subseteq T_x W_{\alpha(i)}$, such that $T\kappa_n^{\alpha(i)}(N_x) \subseteq \operatorname{dom}\exp_n$ holds and $\exp_n$ restricts to a smooth embedding on this set (cf. Lemma D.0.6). By Construction 6.1.6 for $(t\sigma_{\alpha(i)} \diamond_i s\tau_{\alpha(i)})|_{\Omega_{\frac{5}{4},K_{5,i}}}$ and each chart $V^n_{\frac{5}{4},\alpha(i)}$ the local identity (D.0.13) holds. Using the notation introduced, the identity (D.0.13) yields the following formula for $x \in B_{\frac{5}{4}}(0)$:

$$\begin{aligned}
&T\kappa_n^{\alpha(i)} t\sigma_{\alpha(i)} \diamond_i s\tau_{\alpha(i)} (\kappa_n^{\alpha(i)})^{-1}(x) = (x, (t\sigma_{\alpha(i)})_{[n]} \diamond_{[n]} (s\tau_{\alpha(i)})_{[n]}(x)) \\
=&(x, (\exp_n|_{T\kappa_n^{\alpha(i)}(N_x)})^{-1} \exp_n(\exp_n(x, (s\tau_{\alpha(i)})_{[n]}(x)), (t\sigma_{\alpha(i)})_{[n]}(\exp_n(s\tau_{\alpha(i)})_{[n]}(x)))) \\
=&(x, (\exp_n|_{T\kappa_n^{\alpha(i)}(N_x)})^{-1} \exp_n(\exp_n(x, s(\tau_{\alpha(i)})_{[n]}(x)), t(\sigma_{\alpha(i)})_{[n]}(\exp_n s(\tau_{\alpha(i)})_{[n]}(x)))).
\end{aligned}$$

Apply pr_2 to the formula above, to obtain the desired identity for $(t\sigma_{\alpha(i)} \diamond_i s\tau_{\alpha(i)})_{[n]}$. To simplify the notation we introduce the abbreviations $X := (\sigma_{\alpha(i)})_{[n]}$ and $Y := (\tau_{\alpha(i)})_{[n]}$. Recall the following properties of $\exp_n$ (cf. [39, Theorem 1.6.12]): $\exp_n(x,0) = x, d\exp_n(x,0) = \mathrm{id}_{T_x B_{\frac{5}{4}}(0)}$, $\forall x \in B_{\frac{5}{4}}(0)$.
Since $\exp_n$ is injective on $(x,0) \in T\kappa_n^{\alpha(i)}(N_x)$ with $\exp_n(x,0) = x$ and $d\exp_n(x,0) = \mathrm{id}_{T_x B_{\frac{5}{4}}(0)}$, we derive

$$d(\exp_n|_{T\kappa_n^{\alpha(i)}(N_x)})^{-1}(x,\cdot) = \mathrm{id}_{T_x B_{\frac{5}{4}}(0)}$$

For $x \in B_{\frac{5}{4}}(0)$ the facts collected above allow us to obtain

$$\begin{aligned}
&\frac{\partial^2}{\partial s \partial t}\bigg|_{t,s=0} (t\sigma_{\alpha(i)} \diamond_i s\tau_{\alpha(i)})_{[n]}(x)\\
&= \frac{\partial^2}{\partial s \partial t}\bigg|_{t,s=0} (\exp_n|_{T\kappa_n^{\alpha(i)}(N_x)})^{-1} \exp_n(\exp_n(x, sY(x)), tX(\exp_n(x, sY(x))))\\
&= \frac{\partial}{\partial s}\bigg|_{s=0} d(\exp_n|_{T\kappa_n^{\alpha(i)}(N_x)})^{-1} \frac{\partial}{\partial t}\bigg|_{t=0} \exp_n(\exp_n(x, sY(x)), tX(\exp_n(x, sY(x))))\\
&= \frac{\partial}{\partial s}\bigg|_{s=0} d(\exp_n|_{T\kappa_n^{\alpha(i)}(N_x)})^{-1}(\exp_n(x, sY(x)), X(\exp_n(x, sY(x)))).
\end{aligned} \tag{6.3.1}$$

The map $d(\exp_n|_{T\kappa_n^{\alpha(i)}(N_x)})^{-1}$ is linear in the second argument. Hence the rule on partial derivatives (2.1.1) applied to (6.3.1) yields the the following identity:

$$\begin{aligned}
\frac{\partial^2}{\partial s \partial t}\bigg|_{t,s=0} (t\sigma_{\alpha(i)} \diamond_i s\tau_{\alpha(i)})_{[n]}(x) =& d(\exp_n|_{T\kappa_n^{\alpha(i)}(N_x)})^{-1}(\exp_n(x,0), dX(\frac{\partial}{\partial s}\bigg|_{s=0} \exp_n(x, sY(x))))\\
&+ d^{(2)}(\exp_n|_{T\kappa_n^{\alpha(i)}(N_x)})^{-1}(\frac{\partial}{\partial s}\bigg|_{s=0} \exp_n(x, sY(x)), X(\exp_n(x,0)))\\
=& dX(x, Y(x)) + \underbrace{d^{(2)}(\exp_n|_{T\kappa_n^{\alpha(i)}(N_x)})^{-1}(x, Y(x), X(x)))}_{S_{XY}:=}
\end{aligned}$$

The derivative $d^{(2)}(\exp_n|_{T\kappa_n^{\alpha(i)}(N_x)})^{-1}(x,\cdot,\cdot)$ is a symmetric bilinear map by [22, Proposition 1.13]. Hence S_{XY} is symmetric in X and Y. An analogous computation yields:

$$\frac{\partial^2}{\partial s \partial t}\bigg|_{t,s=0} (t\sigma_{\alpha(i)} \diamond_i s\tau_{\alpha(i)})_{[n]}(x) = dY(x, X(x)) + S_{XY}$$

As $C^\infty(\kappa_n^{\alpha(i)}, \mathbb{R}^d)$ is an isomorphism of topological vector spaces, $((\mathrm{ad}([\hat\sigma])[\hat\tau])_{\alpha(i)})_{[n]}$ is given by

$$\begin{aligned}
((\mathrm{ad}([\hat\sigma])[\hat\tau])_{\alpha(i)})_{[n]} &= \frac{\partial^2}{\partial s \partial t}\bigg|_{t,s=0} (t\sigma_{\alpha(i)} \diamond_i s\tau_{\alpha(i)})_{[n]} - \frac{\partial^2}{\partial s \partial t}\bigg|_{t,s=0} (t\tau_{\alpha(i)} \diamond_i s\sigma_{\alpha(i)})_{[n]}\\
&= dX(Y(x)) - dY(X(x)) = d(\sigma_{\alpha(i)})_{[n]}(\tau_{\alpha(i)})_{[n]} - d(\tau_{\alpha(i)})_{[n]}(\sigma_{\alpha(i)})_{[n]}.
\end{aligned}$$

Recall from [50, Defintion I.3.6] that the Lie bracket of vector fields V, W in $\mathfrak{X}\left(\Omega_{\frac{5}{4},K_{5,i}}\right)$ is the unique vector field $[V,W]_i$ such that for each chart $(V^n_{\frac{5}{4},\alpha(i)}, \kappa_n^{\alpha(i)}) \in \mathcal{F}_5(K_{5,i})$ the identity

$$([V,W]_i)_{[n]} = dW_{[n]}V_{[n]} - dV_{[n]}W_{[n]}$$

is satisfied. By the above computation, the negative of the Lie bracket of the vector fields $\sigma_{\alpha(i)}$ and $\tau_{\alpha(i)}$ coincides with $(\mathrm{ad}([\hat\sigma])[\hat\tau])_{\alpha(i)}$ on $\Omega_{\frac{5}{4},K_{5,i}}$. Since $U_i \subseteq \Omega_{\frac{5}{4},K_{5,i}}$ holds, the canonical lift $(\mathrm{ad}([\hat\sigma])[\hat\tau])_i$ on U_i coincides with the negative of the Lie bracket of the canonical lifts of σ_i and τ_i.

By abuse of notation let $[\sigma_i, \tau_i]$ be the Lie bracket of the lifts in $\mathfrak{X}(U_i)$. The families $\{\sigma_i\}_{i\in I}$ and $\{\tau_i\}_{i\in I}$ are families of canonical lifts of the orbisections $[\hat{\sigma}]$ and $[\hat{\tau}]$ with respect to the atlas $\mathcal{A}$. Hence each pair of lifts σ_i, σ_j (respectively τ_i, τ_j) for $i, j \in I$ is ϕ-related for $\phi \in \mathcal{C}h_{U_i,U_j}$ (i.e. (4.2.3) holds). By [46, Assertion 4.6] $[\sigma_i, \tau_i]$ and $[\sigma_j, \tau_j]$ are ϕ-related for each $\phi \in \mathcal{C}h_{U_i,U_j}$ and every pair $i, j \in I$. Hence the family $\{[\sigma_i, \tau_i]\}_{i\in I}$ is a family of canonical lifts for the compactly supported orbisection $\mathrm{ad}([\hat{\sigma}])[\hat{\tau}]$. The result of this section may now be summarized as follows:

6.3.1 Theorem (Lie algebra of $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$) *Identify $T_{\mathrm{id}_{(Q,\mathcal{U})}}\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ via $T_{\mathbf{0}_{Orb}}E$ with the space $\mathfrak{X}_{Orb}(Q)_c$ and the Lie Algebra of $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ with $(\mathfrak{X}_{Orb}(Q)_c, [\cdot,\cdot])$. The Lie bracket $[\cdot,\cdot]$ is defined as follows:*
For arbitrary $[\hat{\sigma}], [\hat{\tau}] \in \mathfrak{X}_{Orb}(Q)_c$, their Lie bracket $[[\hat{\sigma}], [\hat{\tau}]]$ is the unique compactly supported orbisection whose canonical lift on an orbifold chart (U, G, φ) is the negative of the Lie bracket in $\mathfrak{X}(U)$ of their canonical lifts σ_U and τ_U.

If the orbifold is trivial (i.e. a manifold), Theorem 6.3.1 specializes to the well known description of the Lie algebra for the diffeomorphism group of a manifold (cf. [50, Example II.3.14]).

6.4. Regularity properties of $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$

In this section we prove that $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ is a regular Lie group in the sense of Milnor (cf. [46, Definition 7.6]). Unless stated otherwise the notation from Section 6.1 and Section 6.2 will be used. However before further studying this section, the reader should recall the definition of C^k-regularity as outlined in Appendix C.5. The philosophy in the proof of the Lie group properties for $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ has been to compute the relevant data locally on orbifold chart. Hence we investigate the situation on orbifold charts, where we study the flows of vector fields and their differentiability properties. Several facts from the calculus of $C^{r,s}$-mappings (see Definition 2.1.5, cf. [2]) are needed. Crucial to our discussion of regularity is the following differential equation:

6.4.1 Define $f\colon [0,1] \times B_5(0) \times C^r([0,1], C^\infty(B_5(0), \mathbb{R}^d)) \to \mathbb{R}^d$, via $f(t, x, \gamma) := \gamma^\wedge(t, x) := \gamma(t)(x)$ for $r \in \mathbb{N}_0 \cup \{\infty\}$. Consider the evaluation maps $\varepsilon\colon C^\infty(B_5(0), \mathbb{R}^d) \times B_5(0) \to \mathbb{R}^d, \varepsilon(\sigma, x) := \sigma(x)$ and $\varepsilon_1\colon C^r([0,1], C^\infty(B_5(0), \mathbb{R}^d)) \times [0,1] \to C^\infty(B_5(0), \mathbb{R}^d), (\gamma, t) \mapsto \gamma(t)$. By [2, Proposition 3.20], ε is smooth and ε_1 is of class $C^{\infty,r}$. We may rewrite the map f as $f(t, x, \gamma) = \varepsilon(\varepsilon_1(\gamma, t), x)$. Hence the chain rule [2, Lemma 3.17] implies that f is of class $C^{r,\infty}$ with respect to the product $[0,1] \times \big(B_5(0) \times C^r([0,1], C^\infty(B_5(0), \mathbb{R}^d))\big)$. Thus the initial value problem

$$\begin{cases} x'(t) &= f(t, x(t), \gamma) = \gamma^\wedge(t, x(t)), \\ x(t_0) &= x_0, \quad x_0 \in B_5(0) \end{cases} \tag{6.4.1}$$

admits a unique solution $\varphi_{t_0,x_0,\gamma}$ by [2, Theorem 5.6]. Fixing $t_0 = 0$, the flow of (6.4.1)

$$\mathrm{Fl}^f_0 := \mathrm{Fl}^f(0,\cdot)\colon [0,1] \times \big(B_5(0) \times C^r([0,1], C^\infty(B_5(0), \mathbb{R}^d))\big) \supseteq \Omega_0 \to \mathbb{R}^d, (t, (x_0, \gamma)) \mapsto \varphi_{0,x_0,\gamma}(t)$$

is of class $C^{r+1,\infty}$ on the open subset Ω_0 by [2, Proposition 5.9].

6.4.2 Lemma *Let* $r \in \mathbb{N}_0 \cup \{\infty\}$, $\gamma \in C^r([0,1], C^\infty(B_5(0), \mathbb{R}^d))$ *and consider* f *as in 6.4.1.*

(a) *If* γ *satisfies* $\|\gamma(t)\|_{\overline{B_4(0)},0} \leq 1$ *for all* $t \in [0,1]$ *the map* $\mathrm{Fl}_0^f(\cdot, \gamma)$, *is defined on* $[0,1] \times B_3(0)$ *and* $\mathrm{Fl}_0^f([0,1] \times B_3(0) \times \{\gamma\}) \subseteq B_4(0)$ *holds.*

(b) *Consider* $\zeta > 0$ *and a compact subset* $K \subseteq B_3(0)$. *There exists* $0 < \tau \leq 1$ *such that for all* $\gamma \in C^r([0,1], C^\infty(B_5(0), \mathbb{R}^d))$ *with* $\sup_{t\in[0,1]} \|\gamma(t)\|_{\overline{B_4(0)},1} < \tau$ *(cf. Definition C.2.5), we have* $\|\mathrm{Fl}_0^f(t, \cdot, \gamma) - \mathrm{id}_{B_3(0)}\|_{K,1} < \zeta$, *for all* $t \in [0,1]$.

(c) *For* τ *as in* (b) *and* $B_\tau(0) := \left\{ f \in C^\infty(B_5(0), \mathbb{R}^d) \middle| \|f\|_{\overline{B_4(0)},0} < \tau \right\}$, *we obtain a smooth map*

$$F\colon C^r([0,1], B_\tau(0)) \to C^{r+1}([0,1], C^\infty(B_3(0), \mathbb{R}^d)), \gamma \mapsto \mathrm{Fl}_0^f(\cdot, \gamma)|_{[0,1]\times B_3(0)}.$$

Proof. (a) For $x_0 \in B_3(0)$ the maximal solution to the initial value problem (6.4.1) is the mapping $\mathrm{Fl}_0^f(\cdot, x_0, \gamma)$. We claim that it is defined at least on $[0,1]$. Restricting Fl_0^f, we obtain the maximal solution to the initial value problem (6.4.1), whose image remains inside of $B_4(0)$: Denote this solution by $u\colon [0, t_0[\to B_4(0)$. Then u is of class C^1. If $t_0 < 1$ holds, we deduce $\|u(t)\| \leq \|u(0)\| + \|\int_0^{t_0} \gamma^\wedge(t, u(t))dt\| \leq \|x_0\| + 1 =: \rho < 4$ from the Fundamental Theorem of Calculus [22, Theorem 1.5]. Therefore $u|_{[0,t_0[}$ does not leave the compact subset $\overline{B_\rho(0)} \subseteq B_4(0)$. The right hand side of the differential equation (6.4.1) is defined on an open subset of a finite dimensional Banach space, whence by [28, Lemma 3.11] C^k maps coincide with the differentiable maps considered in [43]. One may therefore apply [43, IV. Thm. 2.3.]: The maximal solution must be defined on an intervall strictly larger than $[0, t_0[$, thus contradicting the choice of t_0. We conclude that $\mathrm{Fl}_0^f(\cdot, \gamma)$ maps $[0,1] \times B_3(0)$ into $B_4(0)$.

(b) Observe that $\mathrm{Fl}_0^f(\cdot, \gamma)$ is of class $C^{r+1,\infty}$ by 6.4.1. By [2, Lemma 3.15] $\mathrm{Fl}_0^f(\cdot, \gamma)$ is a C^1-mapping, whence the derivatives required for $\|\cdot\|_{K,1}$ exist. The mapping $h\colon [0,1] \times B_3(0) \to \mathbb{R}^d, h(t,x) := \gamma^\wedge(t, \mathrm{Fl}_0^f(t, x, \gamma))$ is of class $C^{r,\infty}$ by the chain rule [2, Lemma 3.19]. Fix $x \in B_3(0)$ and consider the map $g\colon [0,1] \to \mathcal{L}\left(\mathbb{R}^d\right), g(t) := d_2 \mathrm{Fl}_0^f(t, x, \gamma; \cdot)$. Schwarz' Theorem [2, Proposition 3.6 and Remark 3.7] implies that g is a solution to

$$\begin{cases} y'(t) &= d_2\gamma^\wedge(t, \mathrm{Fl}_0^f(t, x, \gamma); y(t)) \\ y(0) &= \mathrm{id}_{\mathbb{R}^d} \end{cases}. \tag{6.4.2}$$

The domain of $\gamma^\wedge(t, \cdot)$ is an open subset of $\mathbb{R}^d$. Hence the derivative $d_2\gamma^\wedge(t, x; \cdot)$ is determined by the Jacobian matrix. As all norms on $\mathbb{R}^d$ are equivalent, there is a constant $C > 0$, depending only on d and the norm, such that $\|d_2\gamma^\wedge(t, x; \cdot)\|_{\mathrm{op}} \leq C \sup_{|\alpha|=1} \|\partial^\alpha \gamma^\wedge(t, x)\|$ holds. Furthermore $\mathrm{Fl}_0^f(\cdot, \gamma)$ maps $[0,1] \times B_3(0)$ into $B_4(0)$ by (a) and $\|\cdot\|_{\overline{B_4(0)},1}$ controls the partial derivatives. Hence the above estimate yields

$$\sup_{t\in[0,1]} \left\| d_2\gamma^\wedge(t, \mathrm{Fl}_0^f(t, x, \gamma); \cdot) \right\|_{\mathrm{op}} \leq \sup_{t\in[0,1]} C \|\gamma(t)\|_{\overline{B_4(0)},1}.$$

Vice versa, there is a constant $c > 0$, depending only on the Norm and d, such that the following holds

$$\sup_{t\in[0,1]} \sup_{|\alpha|=1} \left\| \partial^\alpha (\mathrm{Fl}_0^f(t, \cdot, \gamma) - \mathrm{id}_{\mathbb{R}^d})(x) \right\| \leq \sup_{t\in[0,1]} c \|g(t) - g(0)\|_{\mathrm{op}}.$$

Let $\rho > 0$ be an upper bound for $\sup_{t\in[0,1]} C\,\|\gamma(t)\|_{\overline{B_4(0)},1}$. The mapping g is of class C^1, whence the Fundamental Theorem of Calculus [22, Theorem 1.5] yields:

$$\begin{aligned}\|g(t) - \mathrm{id}_{\mathbb{R}^d}\|_{\mathrm{op}} = \|g(t) - g(0)\|_{\mathrm{op}} &= \left\|\int_0^t d_2\gamma^\wedge(s, \mathrm{Fl}_0^f(s,x,\gamma), g(s))ds\right\|_{\mathrm{op}} \\ &\leq \int_0^t \rho\,\|g(s)\|_{\mathrm{op}}\,ds = \int_0^t \rho(\|g(0)\|_{\mathrm{op}} + (\|g(s)\|_{\mathrm{op}} - \|g(0)\|_{\mathrm{op}})) \\ &\leq \int_0^t \rho\,\|\mathrm{id}_{\mathbb{R}^d}\|_{\mathrm{op}}\,ds + \int_0^t \rho\,\|g(s) - \mathrm{id}_{\mathbb{R}^d}\|_{\mathrm{op}}\,ds = \rho t + \int_0^t \rho\,\|g(s) - \mathrm{id}_{\mathbb{R}^d}\|_{\mathrm{op}}\end{aligned}$$

Apply Gronwall‘s inequality [3, 6.1 Gronwall's Lemma]) to choose $1 > \frac{\tau_1}{C} > 0$, such that $\sup_{t\in[0,1]} \|\gamma(t)\|_{\overline{B_4(0)},1} < \frac{\tau_1}{C}$ implies

$$\sup_{t\in[0,1]}\sup_{|\alpha|=1} \left\|\partial^\alpha(\mathrm{Fl}_0^f(t,\cdot,\gamma) - \mathrm{id}_{\mathbb{R}^d})(x)\right\| \leq \sup_{t\in[0,1]} c\,\|g(t) - g(0)\|_{\mathrm{op}} < \zeta. \tag{6.4.3}$$

Observe that the estimate (6.4.3) holds for each $x \in B_3(0)$, as the constants did not depend on x. We have to obtain an estimate for Fl_0^f: The set $\overline{B_4(0)} \subseteq \mathbb{R}^d$ is convex, whence the Fundamental Theorem of Calculus [22, Thm. 1.5] with equation (6.4.1) yields for $x \in B_3(0)$:

$$\left\|\mathrm{Fl}_0^f(t,x,\gamma) - \mathrm{id}_{B_3(0)}(x)\right\| = \left\|\mathrm{Fl}_0^f(t,x,\gamma) - \mathrm{Fl}_0^f(0,x,\gamma)\right\| = \left\|\int_0^1 \gamma^\wedge(st, \mathrm{Fl}_0^f(st,x,\gamma))ds\right\|.$$

Set $\sup_{t\in[0,1]} \|\gamma(t)\|_{B_4(0),0} < \zeta$ to obtain $\sup_{t\in[0,1]} \left\|\mathrm{Fl}_0^f(t,x,\gamma) - \mathrm{id}_{B_3(0)}(x)\right\| < \zeta$. The estimates show that $\tau := \min\left\{\zeta, \frac{\tau_1}{C}\right\}$ is a constant with the desired properties.

(c) Let $r \in \mathbb{N}_0 \cup \{\infty\}$, X be a Fréchet-space and $U \subseteq \mathbb{R}^d$ an open subset. By Remark C.2.3, each of the topological spaces $[0,1], C^r([0,1],X), C^r(U,X)$ is a Fréchet space. The set $C^r([0,1], B_\tau(0))$ is an open subset of the Fréchet space $C^r([0,1], C^\infty(B_5(0),\mathbb{R}^d))$ (cf. [23, Lemma 3.6]), whence metrizable. Therefore each finite cartesian product of these spaces is a k-space by [19, XI. 9.3] and we may use the Exponential Law for $C^{r,s}$-maps (cf. [2, Theorem 3.28 (e)]):
Since $\mathrm{Fl}_0^f(\cdot,\gamma)$ is of class $C^{r+1,\infty}$, we deduce that $F(\gamma)$ is in $C^{r+1}([0,1], C^\infty(B_3(0),\mathbb{R}^d)$. Hence F is well-defined and we claim that F is smooth. By [2, Theorem 3.28 (e)] F will be smooth if and only if the following map is a $C^{\infty,r+1}$-mapping:

$$F^\wedge\colon C^r([0,1], B_\tau(0)) \times [0,1] \to C^\infty(B_3(0),\mathbb{R}^d), (\gamma,t) \mapsto F(\gamma)(t).$$

Using [2, Corollary 3.8] and the Exponential Law, this map will be $C^{\infty,r+1}$ if and only if $(F^\wedge)^\vee\colon [0,1] \to C^\infty(C^r([0,1],B_\tau(0)), C^\infty(B_3(0),\mathbb{R}^d)), t \mapsto F(\cdot)(t)$ is of class C^{r+1}. Combine the Exponential Law with [2, Lemma 3.22] to establish the isomorphism

$$\Phi\colon C^\infty(C^r([0,1],B_\tau(0)), C^\infty(B_3(0),\mathbb{R}^d)) \to C^\infty(C^r([0,1],B_\tau(0)) \times B_3(0),\mathbb{R}^d), f \mapsto f^\wedge.$$

Hence $(F^\wedge)^\vee$ will be a C^{r+1}-map if and only if the composition $\Phi \circ (F^\wedge)^\vee$, given by

$$(\mathrm{Fl}_0^f)^\vee\colon [0,1] \to C^\infty(C^r([0,1],B_\tau(0)) \times B_3(0),\mathbb{R}^d), t \mapsto \mathrm{Fl}_0^f(t,\cdot)|_{C^r([0,1],B_\tau(0))\times B_3(0)},$$

is of class C^{r+1}. By the Exponential Law shows that $(\mathrm{Fl}_0^f)^\vee$ is a C^{r+1}-mapping if and only if Fl_0^f of class $C^{r+1,\infty}$ on $[0,1]\times(B_3(0)\times C^r([0,1],B_\tau(0)))$. Summing up F is smooth, since Fl_0^f is of class $C^{r+1,\infty}$.

□

To prove the regularity of $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$, we have to construct a smooth evolution map $\mathfrak{X}_{\mathrm{Orb}}(Q)_c \to \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$. We will assure the smoothness of all relevant maps via patched mapping arguments. These are prepared by following preliminary Lemma.

6.4.3 Lemma *Consider $r\in\mathbb{N}_0\cup\{\infty\}$ and define for $\gamma\in C^r([0,1],\mathfrak{X}(W_{\alpha(i)}))$ and $(V^n_{5,\alpha(i)},\kappa_n^{\alpha(i)})\in\mathcal{F}_5(K_{5,i})$ the C^r-curves $\gamma_{\kappa_n^{\alpha(i)}} := \theta_{\kappa_n^{\alpha(i)}}\circ\gamma$ (cf. Definition C.3.1) and $\gamma_{[n]} := C^\infty((\kappa_n^{\alpha(i)})^{-1},\mathbb{R}^d)\circ\gamma_{\kappa_n^{\alpha(i)}}$. For each $i\in I$ there is an open C^1-neighborhood $\mathcal{E}^i\subseteq\mathfrak{X}(W_{\alpha(i)})$ of the zero-section, such that the following holds:*

(a) *For $\gamma\in C^r([0,1],\mathcal{E}^i)$ we obtain a map $e(\gamma)\in C^{r+1}([0,1],\mathfrak{X}(\Omega_{2,K_{5,i}}))$, defined locally via*

$$e(\gamma)(t)(x)=(\exp_{W_{\alpha(i)}}|_{N_x})^{-1}\circ(\kappa_n^{\alpha(i)})^{-1}\circ\mathrm{Fl}_0^f(t,\kappa_n^{\alpha(i)}(x),\gamma_{[n]}),\quad (t,x)\in[0,1]\times V^n_{2,\alpha(i)} \tag{6.4.4}$$

for f as in 6.4.1 and N_x as in D.0.6. Furthermore for $S_{\alpha(i)}$ as in Construction 6.1.1 V. and $(t,x)\in[0,1]\times V^n_{2,\alpha(i)}$, the following estimates hold:

$$\exp_{W_{\alpha(i)}}\circ e(\gamma)(t)(x)\in V^n_{3,\alpha(i)}\quad\text{and}\quad e(\gamma)(t)(x)\in B_{\rho_{\alpha(i)}}(0_x,S_{\alpha(i)}). \tag{6.4.5}$$

(b) *For each $\gamma\in C^r([0,1],\mathcal{E}_i)$, $e(\gamma)(0)$ is the zero section in $\mathfrak{X}(\Omega_{2,K_{5,i}})$. If γ is the constant map $\gamma\equiv 0_{W_{\alpha(i)}}$, then $e(\gamma)(t)$ is the zero-section for each $t\in[0,1]$.*

(c) *there are smooth maps*

$$\omega_i\colon C^r([0,1],\mathcal{E}_i)\to C^{r+1}([0,1],\mathfrak{X}(\Omega_{2,K_{5,i}})),\gamma\mapsto e(\gamma)$$
$$\theta_i\colon C^r([0,1],\mathcal{E}_i)\to\mathfrak{X}(\Omega_{2,K_{5,i}}),\gamma\mapsto e(\gamma)(1).$$

Proof. The set $\mathcal{F}_5(K_{5,i})$ is finite, whence by Lemma D.0.6 (a), we choose and fix $\nu>0$ with the following properties: For each $y\in\overline{\Omega_{4,K_{5,i}}}$ the map $\exp_{W_{\alpha(i)}}$ is injective on

$$N_y=\bigcup_{(V^n_{5,\alpha(i)},\kappa_n^{\alpha(i)})\in I_y}(T\kappa_n^{\alpha(i)})^{-1}\left(\left\{\kappa_n^{\alpha(i)}(y)\right\}\times B_\nu(0)\right),$$

where $I_y=\left\{x\in\overline{V^n_{4,\alpha(i)}}\middle|(V^n_{5,\alpha i},\kappa_n^{\alpha(i)})\in\mathcal{F}_5(K_{5,i})\right\}$. Lemma D.0.6 (b) holds for the exponential maps $\exp_n$ associated to the pullback metric on $B_5(0)$ with respect to $\rho_{\alpha(i)}$ and $\kappa_n^{\alpha(i)}$.
Consider $(V^n_{5,\alpha(i)},\kappa_n^{\alpha(i)})\in\mathcal{F}_5(K_{5,i})$. By Lemma D.0.3 there are constants $\varepsilon_n>0, 1>\delta_n>0$, such that $a_n^{\alpha(i)}\colon B_4(0)\times B_{\delta_n}(0)\to B_{\varepsilon_n}(0_x), a(x,y):=\exp_n|^{-1}_{B_{\varepsilon_n}(0_x)}(x+y)$ is a smooth map. Shrinking

ε_n, δ_n, without loss of generality $\varepsilon_n < \min\{R_i, \nu\}$ holds, for the constant R_i from Construction 6.1.1 V.. Recall that $\kappa_n^{\alpha(i)}(V^n_{5,\alpha(i)}) = B_5(0)$ holds, whence by Lemma 6.4.2 (b) there is a constant $0 < \tau_n \leq 1$, such that for $\gamma \in C^r([0,1], C^\infty(B_5(0), \mathbb{R}^d))$ with $\sup_{t\in[0,1]} \|\gamma(t)\|_{\overline{B_4(0)},1} \leq \tau_n$ one has

$$\sup_{t\in[0,1]} \left\|\mathrm{Fl}_0^f(t,\cdot,\gamma) - \mathrm{id}_{B_3(0)}\right\|_{\overline{B_2(0)},1} < \delta_n. \tag{6.4.6}$$

Observe that $\delta_n < 1$ together with (6.4.6) implies $\mathrm{Fl}_0^f(t,\cdot,\gamma)(\overline{B_2(0)}) \subseteq B_3(0)$. Consider the open zero-neighborhood $E_n := \left\{ f \in C^\infty(B_5(0), \mathbb{R}^d) \middle| \|f\|_{\overline{B_4(0)},1} < \tau_n \right\}$ and let

$$\mathcal{E}_n^i := \left\{ \sigma \in \mathfrak{X}\left(\Omega_{5,K_{5,i}}\right) \middle| \sigma_{[n]} := \mathrm{pr}_2 \circ T\kappa_n^{\alpha(i)} \circ \sigma \circ (\kappa_n^{\alpha(i)})^{-1} \in E_n \right\}$$

be the open neighborhood of the zero-section in $\mathfrak{X}\left(\Omega_{5,K_{5,i}}\right)$ induced by E_n. Repeating this construction, we obtain open neighborhoods of the zero map (respectively the zero-section) for each chart in $\mathcal{F}_5(K_{5,i})$. Let $V_i := \bigcap_{\mathcal{F}_5(K_{5,i})} \mathcal{E}_n^i \subseteq \mathfrak{X}\left(\Omega_{5,K_{5,i}}\right)$. We claim that the open zero-neighborhood $\mathcal{E}^i := (\mathrm{res}^{W_{\alpha(i)}}_{\Omega_{5,K_{5,i}}})^{-1}(V_i) \subseteq \mathfrak{X}\left(W_{\alpha(i)}\right)$ satisfies the assertion of the Lemma.

(a) Consider $\gamma \in C^r([0,1], V_i)$ and $(V^n_{5,\alpha(i)}, \kappa_n^{\alpha(i)}) \in \mathcal{F}_5(K_{5,i})$. The map h_n sending $\gamma(t)$ to $\gamma_{[n]}(t)$ for $t \in [0,1]$ is continuous linear by [25, Lemma F.6 and Lemma 4.11]. We deduce from [31, Lemma 1.2] that $(h_n)_*\colon C^r([0,1], \mathfrak{X}\left(\Omega_{5,K_{5,i}}\right)) \to C^r([0,1], C^\infty(B_5(0), \mathbb{R}^d)), \gamma \mapsto \gamma_{[n]}$ is continuous linear. Since $\gamma \in V_i$ holds, we have $\gamma_{\kappa_n^{\alpha(i)}} \in C^r([0,1], E_n)$. By construction (6.4.6) holds, $a_n^{\alpha(i)}$ is smooth and $\mathrm{Fl}_0^f(\cdot,\cdot,\gamma_{[n]})$ a $C^{r+1,\infty}$ mapping by 6.4.1. By the Exponential Law [2, Theorem 3.28 (e)] a map in $C^{r+1}([0,1], C^\infty(B_2(0), \mathbb{R}^d))$ may be defined via

$$e(\gamma)_n(t) := a_n^{\alpha(i)} \circ (\mathrm{id}_{B_2(0)}, \mathrm{Fl}_0^f(t,\cdot,\gamma_{[n]}) - \mathrm{id}_{B_2(0)}), \quad t \in [0,1]. \tag{6.4.7}$$

Observe that $e(\gamma)_n(t)(B_2(0)) \subseteq B_{\varepsilon_n}(0)$ holds for each $t \in [0,1]$. The construction may be repeated for each chart in $\mathcal{F}_5(K_{5,i})$. As $\varepsilon_n < \min\{\nu, R_i\}$ holds, we obtain by definition of ν and R_i for $(t,x) \in [0,1] \times B_2(0)$:

$$T(\kappa_n^{\alpha(i)})^{-1} e(\sigma)_n(t)(x) \in N_{(\kappa_n^{\alpha(i)})^{-1}(x)} \cap B_{\rho_{\alpha(i)}}(0_{(\kappa_n^{\alpha(i)})^{-1}(x)}, S_{\alpha(i)}). \tag{6.4.8}$$

By Lemma D.0.6 (b) the local formula (6.4.4) coincides with (6.4.7). From the uniqueness of the flow $\mathrm{Fl}_0^f(\cdot,\gamma_{[n]})$ we deduce that the mappings $e(\gamma)_n$ coincide on the intersections of their domains, whence we obtain a map $e(\gamma) \in C^{r+1}([0,1], \mathfrak{X}\left(\Omega_{2,K_{5,i}}\right))$. The local representative of this time dependend vector field on $(V^n_{5,\alpha(i)}, \kappa^{\alpha(i)}, n) \in \mathcal{F}_5(K_{5,i})$ is $e(\gamma)_n$.

For $x \in V^n_{2,\alpha(i)}$, the formula of $e(\gamma)_n$ together with Lemma D.0.6 (b) allows us to compute

$$(\exp_{W_{\alpha(i)}}|_{N_x}) \circ e(\gamma)(t)(x) = \kappa_n^{\alpha(i)} \exp_n e(\gamma)_n(t)(\kappa_n^{\alpha(i)}(x)) = \kappa_n^{\alpha(i)} \mathrm{Fl}_0^f(t, \kappa_n(x), \gamma_{[n]}) \in V^n_{3,\alpha(i)}$$

Furthermore (6.4.8) shows that the estimate (6.4.5) holds. The map $\mathrm{res}^{W_{\alpha(i)}}_{\Omega_{5,K_{5,i}}}$ is continuous linear by [25, Lemma F.15], whence $(\mathrm{res}^{W_{\alpha(i)}}_{\Omega_{5,K_{5,i}}})_*\colon C^r([0,1], \mathfrak{X}\left(W_{\alpha(i)}\right)) \to C^r([0,1], \mathfrak{X}\left(\Omega_{5,K_{5,i}}\right))$ is continuous linear by [31, Lemma 1.2]. Assign to $\gamma \in C^r([0,1], \mathcal{E}^i)$ the vector field $e(\mathrm{res}(\gamma))$. By abuse of notation, we will omit res from now on, i.e. for $\gamma \in C^r([0,1], \mathcal{E}^i)$, $e(\gamma) := e(\mathrm{res}(\gamma))$

(b) The map $\mathrm{Fl}_0^f(t, \kappa_n^{\alpha(i)}(x), \gamma_{[n]})$ is a solution to the initial value problem (6.4.1) with initial value $\mathrm{Fl}_0^f(0, \kappa_n^{\alpha(i)}(x), \gamma_{[n]}) = \kappa_n^{\alpha(i)}(x)$. We obtain $e(\gamma)(0)(x) = (\exp_{W_{\alpha(i)}}|_{N_x})^{-1}(x) = 0_x$ from (6.4.4), since $\exp_{W_{\alpha(i)}}(0_x) = x$ holds and on N_x the map $\exp_{W_{\alpha(i)}}$ is injective.

If $\gamma_{[n]} \equiv 0$, its flow is defined as $\mathrm{Fl}_0^f(t, \kappa_n^{\alpha(i)}(x), 0) = \kappa_n^{\alpha(i)}(x)$. Analogous to the previous argument, $e(\gamma)(t)$ is the zero section for each $t \in [0,1]$.

(c) We prove the smoothness of ω^i, δ^i via a patched mapping argument. To this end consider the continuous linear maps $p_n^s\colon \mathfrak{X}\left(\Omega_{s,K_{5,i}}\right) \to C^\infty(B_s(0), \mathbb{R}^d), \sigma \mapsto \sigma_{\kappa_n^{\alpha(i)}} \circ (\kappa_n^{\alpha(i)})^{-1}|_{B_s(0)}$ for $s \in [1,5]$. By Definition C.3.1 $p^s := (p_n^s)_{(V^n_{5,\alpha(i)}, \kappa_n^{\alpha(i)}) \in \mathcal{F}_5(K_{5,i})}$ is a topological embedding with closed image. Thus Lemma C.3.5 yields a topological embedding with closed image

$$p_*^s\colon C^r([0,1], \mathfrak{X}\left(\Omega_{s,K_{5,i}}\right)) \to \bigoplus_{\mathcal{F}_5(K_{5,i})} C^r([0,1], C^\infty(B_s(0), \mathbb{R}^d)), \gamma \mapsto (p_n^s \circ \gamma)_{\mathcal{F}_{5,K_{5,i}}}.$$

Consider the maps $h^i\colon C^r([0,1], V_i) \to C^{r+1}([0,1], \mathfrak{X}\left(\Omega_{5,K_{5,i}}\right)), \gamma \mapsto e(\gamma)$. We claim that there are smooth maps D_n such that the following diagram becomes commutative:

$$\begin{tikzcd}
C^r([0,1], \mathcal{E}_i) \arrow[r, "\mathrm{res}"] & C^r([0,1], V_i) \arrow[r, "h^i"] \arrow[d, "p_*^5"'] & C^{r+1}([0,1], \mathfrak{X}\left(\Omega_{2,K_{5,i}}\right)) \arrow[d, "p_*^2"] \\
& \bigoplus_{\mathcal{F}_5(K_{5,i})} C^r([0,1], E_n) \arrow[r, dotted, "(D_n)_{\mathcal{F}_5(K_{5,i})}"] & \bigoplus_{\mathcal{F}_5(K_{5,i})} C^{r+1}([0,1], C^\infty(B_2(0), \mathbb{R}^d))
\end{tikzcd}$$

Observe that the vertical arrows are given by embeddings with closed image and composition in the upper row yields $\omega_i = h^i \circ \mathrm{res}$. Since res is a smooth map, ω_i will be smooth if h^i is smooth. If the claim were true, then by Proposition C.3.7 h^i and thus ω_i will be smooth. Consider the open sets $\lfloor \overline{B_2(0)}, B_{\delta_n}(0) \rfloor_\infty \subseteq C^\infty(B_3(0), \mathbb{R}^d)$ and define

$$(a_n^{\alpha(i)})_*\colon \lfloor \overline{B_2(0)}, B_\delta(0) \rfloor_\infty \to C^\infty(B_2(0), \mathbb{R}^d), \quad (a_n^{\alpha(i)})_*(g)(x) := a_n^{\alpha(i)}(x, g(x)).$$

By [25, Proposition 4.23 (a)] $(a_n^{\alpha(i)})_*$ is smooth, since $a_n^{\alpha(i)}$ is smooth. From Lemma 6.4.2 and the definition of E_n we deduce that $F_n\colon C^r([0,1], E_n) \to C^{r+1}([0,1], C^\infty(B_3(0), \mathbb{R}^d))$, $F_n(\gamma)(t) := \mathrm{Fl}_0^f(t, \cdot, \gamma)|_{B_3(0)} - \mathrm{id}_{B_3(0)}$, $t \in [0,1]$ is smooth. The estimate (6.4.6), yields $F_n(\gamma)([0,1]) \subseteq \lfloor \overline{B_2(0)}, B_{\delta_n}(0) \rfloor_\infty$. Thus $(a_n^{\alpha(i)})_* \circ F_n^\wedge\colon [0,1] \times C^r([0,1], E_n) \to C^\infty(B_2(0), \mathbb{R}^d)$ is a $C^{r+1,\infty}$-map by the Exponential Law [2, Theorem 3.28 (e)] and [2, Lemma 3.18]. Apply [2, Corollary 3.8 and Theorem 3.28 (e)] to obtain a smooth map:

$$D_n\colon C^r([0,1], E_n) \to C^{r+1}([0,1], C^\infty(B_2(0), \mathbb{R}^d), \gamma \mapsto ((a_n^{\alpha(i)})_* \circ F_n^\wedge)^\vee(\gamma) = (a_n^{\alpha(i)})_* \circ F_n(\gamma)$$

with $D_n(0) = 0$. A computation with (6.4.4) and Lemma D.0.6 (b) shows that $(D_n)_{\mathcal{F}_5(K_{5,i})}$ makes the above diagramm commutative. By [2, Prop. 2.20] we consider the smooth evaluation $\varepsilon_1\colon C^{r+1}([0,1], \mathfrak{X}\left(\Omega_{2,K_{5,i}}\right)) \to \mathfrak{X}\left(\Omega_{2,K_{5,i}}\right), \gamma \mapsto \gamma(1)$. Since $\theta_i = \varepsilon_1 \circ \omega_i$ holds, θ_i is smooth.

□

6.4.4 Lemma *In the setting of Lemma 6.4.3 define the open set* $\mathcal{E} := \Lambda_{\mathcal{C}}^{-1}(\bigoplus_{i\in I}\mathcal{E}^i) \subseteq \mathfrak{X}_{Orb}(Q)_c$*, where* $\mathcal{C}$ *is the orbifold atlas introduced in 6.1.3. Let* $r \in \mathbb{N}_0 \cup \{\infty\}$*. For each* $i \in I$ *and* $\gamma \in C^r([0,1], \mathfrak{X}_{Orb}(Q)_c)$*, we define* $\gamma_{\alpha(i)} \colon [0,1] \to \mathfrak{X}(W_{\alpha(i)}), t \mapsto (\gamma(t))_{\alpha(i)}$*, where* $(\gamma(t))_{\alpha(i)}$ *is the canonical lift of* $\gamma(t)$ *with respect to the chart* $(W_{\alpha(i)}, H_{\alpha(i)}, \varphi_{\alpha(i)})$*.*

(a) *If* $\gamma \in C^r([0,1], \mathfrak{X}_{Orb}(Q)_c)$ *holds, the map* $\gamma_{\alpha(i)}$ *is of class* C^r *and for* $i \in I$*, the map* $p_i \colon C^r([0,1], \mathfrak{X}_{Orb}(Q)_c) \to C^r([0,1], \mathfrak{X}(W_{\alpha(i)})), \gamma \mapsto \gamma_{\alpha(i)}$ *is continuous linear.*
(b) *For each* $\gamma \in C^r([0,1], \mathcal{E})$*, we obtain a path* $e(\gamma) \in C^{r+1}([0,1], \mathfrak{X}_{Orb}(Q)_c)$ *whose canonical lifts with respect to* $\mathcal{A}$ *are given by* $e(p_i(\gamma))|_{U_i}, i \in I$*.*

Proof. (a) Pick $\gamma \in C^r([0,1], \mathfrak{X}_{\mathrm{Orb}}(Q)_c)$. By construction $\Lambda_{\mathcal{C}} \circ \gamma \in C^r([0,1], \bigoplus_{i\in I}\mathfrak{X}(W_{\alpha(i)}))$ is a compact set. Arguing as in the proof of Lemma C.3.5, γ induces a family of maps $(\gamma_{\alpha(i)})_{i\in I} \in \bigoplus_{i\in I} C^r([0,1], \mathfrak{X}(W_{\alpha(i)}))$. Recall from the Definition 4.3.3 of the c.s. orbisection topology that each map $\tau_{W_{\alpha(i)}} \colon \mathfrak{X}_{\mathrm{Orb}}(Q)_c \to \mathfrak{X}(W_{\alpha(i)}), [\hat{\sigma}] \mapsto \sigma_{W_{\alpha(i)}}$ is continuous linear. By [31, Lemma 1.2] p_i is a continuous linear map, as $p_i = (\tau_{W_{\alpha(i)}})_*$ holds.

(b) Consider the family of time-dependent vector fields $(e(\gamma_{\alpha(i)})|_{U_i}(s))_{i\in I}$ constructed in Lemma 6.4.3 (a). We claim that for fixes $s \in [0,1]$, these vector fields are a canonical family of lifts of an orbisection. It is sufficient to check the following stronger condition:
For each $i, j \in I$ *and any change of charts map* $\mu \colon \Omega_{2,K_{5,i}} \supseteq \operatorname{dom}\mu \to \operatorname{cod}\mu \subseteq \Omega_{2,K_{5,j}}$*,* $e(\gamma_{\alpha(j)})(s) \circ \mu = T\mu \circ e(\gamma_{\alpha(i)})(s)|_{\operatorname{dom}\mu}$ *holds.*
We check the condition locally: Pick $x \in \operatorname{dom}\mu$ together with charts $(V^n_{5,\alpha(i)}, \kappa^{\alpha(i)}_n) \in \mathcal{F}_5(K_{5,i})$, $(V^m_{5,\alpha(j)}, \kappa^{\alpha(j)}_m) \in \mathcal{F}_5(K_{5,j})$, such that $x \in V^n_{2,\alpha(i)}$ and $\mu(x) \in V^m_{2,\alpha(j)} \subseteq \Omega_{2,K_{5,j}}$ hold. Since $\gamma_{\alpha(i)} \in \mathcal{E}_i$ holds, (6.4.4) and the estimate (6.4.5) yield well-defined maps

$$\varphi_x \colon [0,1] \to V^n_{3,\alpha(i)}, t \mapsto (\kappa^{\alpha(i)}_n)^{-1}\operatorname{Fl}^f_0(t, \kappa^{\alpha(i)}_n(x), \gamma_{\alpha(i)[n]})$$
$$\varphi_{\mu(x)} \colon [0,1] \to V^m_{3,\alpha(j)}, t \mapsto (\kappa^{\alpha(j)}_m)^{-1}\operatorname{Fl}^f_0(t, \kappa^{\alpha(j)}_m(x), \gamma_{\alpha(j)[m]}).$$

These maps are C^1-integral curves for the (time-dependent) vector field $\gamma_{\alpha(i)}$ with initial condition $\varphi_x(0) = x$, respectively for $\gamma_{\alpha(j)}$ with $\varphi_{\mu(x)}(0) = \mu(x)$ (using the terminology of [43, IV, §2]). The charts in $\mathcal{F}_5(K_{5,\alpha(i)})$ are contained in some $Z^{\alpha(i)}_r$ by Construction 6.1.1. Since $x \in \mathcal{K}_{\alpha(i)}$ and $\mu(x) \in \mathcal{K}_{\alpha(j)}$ hold, there is a change of charts $\lambda \colon Z^r_{\alpha(i)} \to W_{\alpha(j)}$ with $\lambda(x) = \mu(x)$. Composing λ with a suitable element of $H_{\alpha(j)}$, without loss of generality there is an open neighborhood U_x of x with $\mu|_{U_x} = \lambda|_{U_x}$. The set $V^n_{5,\alpha(i)}$ is contained in $\operatorname{dom}\lambda$, whence $\lambda \circ \varphi_x \colon [0,1] \to W_{\alpha(j)}$ defines a C^1-curve, such that $\lambda \circ \varphi_x(0) = \lambda(x) = \mu(x) \in \Omega_{2,K_{5,i}}$ holds. For fixed $t \in [0,1]$, the vector fields $\gamma_{\alpha(i)}(t)$ and $\gamma_{\alpha(j)}(t)$ are members of a canonical family of lifts of an orbisection, i.e. $\gamma_{\alpha(j)}(t) \circ \lambda = T\lambda\gamma_{\alpha(i)}(t)|_{\operatorname{dom}\lambda}$ holds. We compute:

$$\gamma_{\alpha(j)}(t)(\lambda\varphi_x(t)) = T\lambda\gamma_{\alpha(i)}(t)(\varphi_x(t)) = T\lambda(\tfrac{\partial}{\partial t}\varphi_x)(t) = \tfrac{\partial}{\partial t}(\lambda \circ \varphi_x)(t)$$

Thus the C^1-curve $\lambda \circ \varphi_x$ is an integral curve for the time-dependent vector field $\gamma_{\alpha(j)}$ with initial condition $\lambda \circ \varphi_x(0) = \lambda(x) = \mu(x)$. On the other hand, the same is true for the C^1 curve $\varphi_{\mu(x)}$. As integral curves for (time-dependent) vector fields are unique (cf. [43, IV. Theorem 2.1] with [43, p. 71]) we derive $\lambda \circ \varphi_x = \varphi_{\mu(x)}$.

Computing locally, we exploit that $\lambda \circ (\kappa_n^{\alpha(i)})^{-1}$ is a Riemannian embedding of $B_5(0)$ into $W_{\alpha(j)}$. In particular by [41, IV. Proposition 2.6] the identity

$$\exp_{W_{\alpha(j)}} T\lambda(\kappa_n^{\alpha(i)})^{-1}(v) = \lambda(\kappa_n^{\alpha(i)})^{-1} \exp_n(v) \quad \forall v \in \operatorname{dom} \exp_n$$

holds. In particular the estimates (6.4.5) and (6.4.8) hold. With Lemma D.0.6 (b) and the identity (6.4.4) for $e(\gamma_{\alpha(i)})$ on $[0,1] \times V^n_{2,\alpha(i)}$, one deduces from the above identity

$$\begin{aligned} \exp_{W_{\alpha(j)}} T\lambda e(\gamma_{\alpha(i)})(s)(x) &= \exp_{W_{\alpha(j)}} T\lambda(T\kappa_n^{\alpha(i)})^{-1} T\kappa_n^{\alpha(i)} e(\gamma_{\alpha(i)})(s)(x) \\ &= \lambda(\kappa_n^{\alpha(i)})^{-1} \exp_n T\kappa_n^{\alpha(i)} e(\gamma_{\alpha(i)})(s)(x) \\ &= \lambda(\kappa_n^{\alpha(i)})^{-1} \kappa_n^{\alpha(i)} \exp_{W_{\alpha(i)}|N_x} e(\gamma_{\alpha(i)})(s)(x) \\ &= \lambda(\kappa_n^{\alpha(i)})^{-1} \mathrm{Fl}_0^f(s, \kappa_n^{\alpha(i)}(x), \gamma_{\alpha(i)[n]}) = \lambda \circ \varphi_x(s) = \varphi_{\mu(x)}(s) \end{aligned}$$

On the other hand, the local formula (6.4.4) for $e(\gamma_{\alpha(j)})$ on $[0,1] \times V^m_{2,\alpha(j)}$ implies

$$\exp_{W_{\alpha(j)}} \circ e(\gamma_{\alpha(j)})(s)(\mu(x)) = \varphi_{\mu(x)}(s) = \exp_{W_{\alpha(j)}} T\lambda e(\gamma_{\alpha(i)})(s)(x).$$

By construction $\lambda(x) = \mu(x) \in \mathcal{K}_j^\circ$ holds. Furthermore the mappings $e(\gamma_{\alpha(j)})(s)$ and $e(\gamma_{\alpha(i)})(s)$ are vector fields which satisfy the estimate (6.4.5). Together with these facts, the definition of the constants (cf. Construction 6.1.1 V.) yields:

$$e(\gamma_{\alpha(j)})(s)(\mu(x)), T\lambda e(\gamma_{\alpha(i)})(s)(x) \in B_{\rho_{\alpha(j)}}(0_{\mu(x)}, s_{\alpha(j)}) \subseteq \hat{O}_{\alpha(j)}.$$

The map $\exp_{W_{\alpha(j)}}$ is injective on the intersection $\hat{O}_{\alpha(j)} \cap T_{\mu(x)} W_{\alpha(j)}$. Hence from the above identity $e(\gamma_{\alpha(j)})(s) \circ \mu(x) = T\mu \circ e(\gamma_{\alpha(i)})(s)(x)$ follows, thus proving the claim. Since U_i is contained in $\Omega_{2,K_{5,i}}$, we deduce that the family $(e(\gamma_{\alpha(i)})|_{U_i})_{i\in I}$ is a canonical family for an orbisection. Thus Remark 4.2.10 (a) shows that this family induces a well defined orbisection $e(\gamma)(s)$. Observe that $\Lambda_{\mathcal{C}} \circ \gamma([0,1])$ factors through a finite subset of $\mathcal{C}$ by [10, III, §1, No. 4, Proposition 5]. We derive from Lemma 6.4.3 (b) that there are only finitely many members of $(e(\gamma_{\alpha(i)})(s))_{i\in I}$ which are not the zero-section. Assume that the finite subset $F \subseteq I$ satisfies $e(\gamma_{\alpha(i)})|_{[0,1]\times U_i} \not\equiv 0_{U_i}$ if and only if $i \in F$. Then $\operatorname{supp}[e(\gamma)(s)] \subseteq \bigcup_{i\in F} \varphi_{\alpha(i)}(W_{\alpha(i)})$. Since each $\varphi_{\alpha(i)}(W_{\alpha(i)})$ is a relatively compact subset of Q, the orbisection $[e(\gamma)(s)]$ is compactly supported.

We are left to prove that the assignment $[0,1] \to \mathfrak{X}_{\mathrm{Orb}}(Q)_c, s \mapsto e(\gamma)(s)$ is of class C^{r+1}. Identify $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$ via $\Lambda_{\mathcal{A}}$ with a sequentially closed subspace of $\bigoplus_{i\in I} \mathfrak{X}(U_i)$. It suffices to prove that $\Lambda_{\mathcal{A}} \circ e(\gamma)$ is contained in $C^{r+1}([0,1], \bigoplus_{i\in I} \mathfrak{X}(U_i))$. The path $\Lambda_{\mathcal{A}} \circ e(\gamma)$ factors through the inclusion $\bigoplus_{i\in F} \mathfrak{X}(U_i) \hookrightarrow \bigoplus_{i\in I} \mathfrak{X}(U_i)$. Each component is given by the C^{r+1}-path $e(p_i(\gamma))|_{U_i}$, whence $\Lambda_{\mathcal{A}} \circ e(\gamma)$ is a path of class C^{r+1} as a map into $\bigoplus_{i\in I} \mathfrak{X}(U_i)$.

□

To assure the smoothness of the evolution map on the Lie group, we exploit the patched locally convex structure of $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$. Unfortunately $C^r([0,1], \mathfrak{X}_{\mathrm{Orb}}(Q)_c)$ will inherit this structure only if $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$ is countably patched (cf. Lemma C.3.5. To assure this condition we require:

Convention: For the rest of this section we let Q be a σ-compact (or second countable) space.

6.4.5 Lemma *Let Q be a σ-compact space and $r \in \mathbb{N}_0$. The maps*

$$\omega\colon C^r([0,1],\mathcal{E}) \to C^{r+1}([0,1],\mathfrak{X}_{Orb}(Q)_c), [\hat{\sigma}] \mapsto [e(\sigma)]$$
$$\mathrm{evol}\colon C^r([0,1],\mathcal{E}) \to \mathfrak{X}_{Orb}(Q)_c, [\hat{\sigma}] \mapsto [e(\sigma)](1)$$

are smooth and map the constant path $\gamma \equiv \mathbf{0}_{Orb}$ to itself respectively to $\mathbf{0}_{Orb}$.

Proof. For $\mathcal{A}$ as in Construction 6.1.1 and $\mathcal{C}$ as in 6.1.3, consider the mappings

$$P_{\mathcal{A}}\colon C^{r+1}([0,1],\mathfrak{X}_{\mathrm{Orb}}(Q)_c) \to \bigoplus_{i\in I} C^{r+1}([0,1],\mathfrak{X}(U_i)), \gamma \mapsto \Lambda_{\mathcal{A}} \circ \gamma = (\gamma_{U_i})_{i\in I}.$$
$$P_{\mathcal{C}}\colon C^{r}([0,1],\mathfrak{X}_{\mathrm{Orb}}(Q)_c) \to \bigoplus_{i\in I} C^{r}([0,1],\mathfrak{X}\left(W_{\alpha(i)}\right)), \gamma \mapsto \Lambda_{\mathcal{C}} \circ \gamma = (\gamma_{\alpha(i)})_{i\in I}.$$

The topological space Q is σ-compact and $\mathcal{A},\mathcal{C}$ are locally finite, whence I is countable. Corollary 4.3.6 (c) shows that the mappings $\Lambda_{\mathcal{A}}, \Lambda_{\mathcal{C}}$ turn $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$ into a patched topological space. As $r < \infty$ holds, an application of Lemma C.3.5 proves: $P_{\mathcal{A}}, P_{\mathcal{C}}$ are linear topological embeddings with closed image, whose components form a patchworks, for $C^{r+1}([0,1],\mathfrak{X}_{\mathrm{Orb}}(Q)_c)$ respectively $C^{r}([0,1],\mathfrak{X}_{\mathrm{Orb}}(Q)_c)$.
The maps ω and evol are well defined by Lemma 6.4.4 (b) and we claim that they are smooth. For $i \in I$ let $\mathrm{res}_{U_i}^{\Omega_{2,K_{5,i}}}\colon \mathfrak{X}\left(\Omega_{2,K_{5,i}}\right) \to \mathfrak{X}(U_i)$ be the restriction map. These mappings are linear and continuous by [25, Lemma F.15 (a)]. Thus $r_i := C^{r+1}([0,1], \mathrm{res}_{U_i}^{\Omega_{2,K_{5,i}}})$ is continuous and linear by [31, Lemma 1.2], whence a smooth map. For $i \in I$ consider the smooth map ω_i defined in Lemma 6.4.3. By Lemma 6.4.3 (b) the smooth map $r_i \circ \omega_i$ maps the constant path $\gamma \equiv O_{W_{\alpha(i)}}$ to the constant path whose image is the zero-section. From the definitions we obtain

$$(r_i \circ \omega_i)_{i\in I} \circ P_{\mathcal{C}}|_{\mathcal{E}}^{\oplus_{i\in I}\mathcal{E}_i} = \Lambda_{\mathcal{A}}\omega. \tag{6.4.9}$$

Hence ω is smooth on the patches and we deduce from (6.4.9) with Proposition C.3.7 that ω is a smooth map. As the evaluation map $\mathrm{ev}_1\colon C^{r+1}([0,1],\mathfrak{X}_{\mathrm{Orb}}(Q)_c) \to \mathfrak{X}_{\mathrm{Orb}}(Q)_c, \gamma \mapsto \gamma(1)$ is smooth (cf. [2, Proposition 3.20]), the smoothness of evol follows from $\mathrm{ev}_1 \circ \omega = \mathrm{evol}$. The last assertion is a direct consequence of Lemma 6.4.3 (b). □

6.4.6 Lemma *Let $\mathcal{H}_\rho \subseteq \mathfrak{X}_{Orb}(Q)_c$ be the open zero-neighborhood of Theorem 6.2.4. Consider an open identity-neighborhood $\mathcal{S} \subseteq E(\mathcal{H}_\rho)$ which is symmetric, i.e. $\mathcal{S} = \mathcal{S}^{-1}$ holds. There is an open subset $\mathbf{0}_{Orb} \in \mathcal{R} \subseteq \mathcal{E} \subseteq \mathfrak{X}_{Orb}(Q)_c$, such that $\omega(C^r([0,1],\mathcal{R})) \subseteq C^{r+1}([0,1],E^{-1}(\mathcal{S}))$ holds.*

Proof. Consider the C^0-neighborhood of the constant path $\gamma_{\mathbf{0}_{\mathrm{Orb}}} \equiv \mathbf{0}_{\mathrm{Orb}}$:

$$C^{r+1}([0,1],E^{-1}(\mathcal{S})) := C^0([0,1],E^{-1}(\mathcal{S})) \cap C^{r+1}([0,1],\mathfrak{X}_{\mathrm{Orb}}(Q)_c).$$

Specializing to $r = 0$ in Lemma 6.4.5, $\omega\colon C^0([0,1],\mathcal{E}) \to C^1([0,1],\mathfrak{X}_{\mathrm{Orb}}(Q)_c)$ is smooth with $\omega(\gamma_{\mathbf{0}_{\mathrm{Orb}}}) = \gamma_{\mathbf{0}_{\mathrm{Orb}}}$. We obtain an open non-empty zero-neighborhood $\omega^{-1}(C^1([0,1],E^{-1}(\mathcal{S}))) \subseteq C^0([0,1],\mathcal{E})$. The defintion of the compact open topology yields an open set $\mathbf{0}_{\mathrm{Orb}} \in \mathcal{R} \subseteq \mathfrak{X}_{\mathrm{Orb}}(Q)_c$, such that $\gamma_{\mathbf{0}_{\mathrm{Orb}}} \in C^0([0,1],\mathcal{R}) \subseteq \omega^{-1}(C^1([0,1],E^{-1}(\mathcal{S})))$ holds. This set is open in $C^r([0,1],\mathcal{E})$ for each $r \in \mathbb{N}_0$ and ω maps it into $C^{r+1}([0,1],E^{-1}(\mathcal{S}))$. □

Observe that by construction we also obtain $\mathrm{evol}(C^r([0,1],\mathcal{R})) \subseteq \mathcal{H}_\rho$. We shall see presently, that with the maps constructed in Lemma 6.4.5, a smooth evolution for the Lie group $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ may be constructed. We would like to apply methods similar to the manifold case (cf. [46, p. 1046]) to prove the regularity of $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$. However, if $(Q,\mathcal{U})$ is a non-trivial orbifold, it is more difficult to verify the existence of right logarithmic derivatives. We need representatives of the orbifold diffeomorphisms in $\mathcal{S}$ taylored to this purpose:

6.4.7 Lemma *Consider* $[\hat{f}] \in \mathcal{S}$ *with* $[\hat{f}] = [\hat{E}^\sigma]$ *for some* $[\sigma] \in \mathcal{H}_\rho$*. For each* $[\hat{g}] \in \mathcal{S}$ *there is a representative* $E_{\hat{f}}(\hat{g})$ *with lifts* $\left\{ E_{\hat{f}}(\hat{g})_i \middle| i \in I \right\}$ *such that the following properties are satisfied:*

(a) *for each* $i \in I$ *the lift* $E_{\hat{f}}(\hat{g})_i$ *is an embedding in* $C^\infty(e^{\sigma_i}(U_i), W_{\alpha(i)})$ *(cf. Lemma 6.1.2),*

(b) *if* $[\hat{g}] = [\hat{f}]^{-1}$ *holds, the lifts are given by* $E_{\hat{f}}(\hat{f}^{-1})_i = (e^{\sigma_i})^{-1}$ *for all* $i \in I$*.*

Proof. Let $[\hat{\tau}^g]$ be the unique preimage of $[\hat{g}]$ with respect to E. From $[\hat{g}] = E([\hat{\tau}^g]) = [\exp_{\mathrm{Orb}}] \circ [\hat{\tau}^g]|^{\Omega}$ we deduce that the claim will hold if there are representative of $[\exp_{\mathrm{Orb}}]$ and $[\hat{\tau}^g]|^{\Omega}$, whose composition yields the desired representative. The map $[\hat{f}]$ is an orbifold diffeomorphism with representative $\widehat{E^\sigma} = (E^\sigma, \{ e^{\sigma_i} | i \in I \}, [P,\nu])$. Hence the orbifold charts $\{ (e^{\sigma_i}(U_i), G_i, \varphi_{\alpha(i)}|_{e^{\sigma_i}(U_i)}) \}_{i\in I}$ (cf. Lemma 6.1.2) cover Q. Recall the following details from the proof of Lemma 6.1.2:
By Step 3, $H_{\alpha(i)}.\,\mathrm{Im}\, e^{\sigma_i} \subseteq \Omega_{2,i}$ is an invariant subset, such that $\mathrm{Im}\, e^{\sigma_i}$ is $H_{\alpha(i)}$-stable. Using Lemma 6.1.2 iii., the canonical lifts $\tau^g_{\alpha(i)}$ map $\mathrm{Im}\, e^{\sigma_i}$ to $\hat{O}_{\alpha(i)}$. Thus $\left\{ \tau^g_{\alpha(i)}|^{O_{\alpha(i)}}_{\mathrm{Im}\, e^{\sigma_i}} \middle| i \in I \right\}$ is a family of lifts for a representative $\hat{\tau}'$ of $[\hat{\tau}^g]|^{\Omega}$. As $\Omega_{2,i} \subseteq \mathcal{K}^\circ_{\alpha(i)}$ holds, we obtain an open subset $T\,\mathrm{Im}\, e^{\sigma_i} \cap \hat{O}_{\alpha(i)} \subseteq \mathrm{dom}\exp_{W_{\alpha(i)}}$ (cf. Construction 6.1.1 IV.). This set is G_i-stable, whence $\exp_{W_{\alpha(i)}}|_{T\,\mathrm{Im}\, e^{\sigma_i} \cap \hat{O}_{\alpha(i)}}$ is a lift of the orbifold exponential map $\exp_{\mathrm{Orb}}$. By Remark 5.2.4 (a) there is a respresentative $\widetilde{\exp_{\mathrm{Orb}}}$ of $[\exp_{\mathrm{Orb}}]$, whose family of lifts contains $\left\{ \exp_{W_{\alpha(i)}}|_{T\,\mathrm{Im}\, e^{\sigma_i} \cap \hat{O}_{\alpha(i)}} \right\}_{i\in I}$. Composing $\widetilde{\exp_{\mathrm{Orb}}}$ and $\hat{\tau}'$ we obtain a representative of $E([\hat{\tau}^g]) = [\hat{g}]$ whose lifts are the smooth mappings

$$E(\hat{f}; \hat{E}^{\tau'})_i := (\exp_{W_{\alpha(i)}}|_{T\,\mathrm{Im}\, e^{\sigma_i} \cap O_{\alpha(i)}}) \circ \tau^g_{\alpha(i)}|^{O_{\alpha(i)}}_{\mathrm{Im}\, e^{\sigma_i}}. \tag{6.4.10}$$

As a consequence of the proof of Lemma 6.1.2, these maps are equivariant smooth embeddings.
Since e^{σ_i} is a lift for $[\hat{f}]$, for each $i \in I$, the map $E(\hat{f}; \hat{f}^{-1})_i \circ (e^{\sigma_i})$ is a change of orbifold charts. Hence for each $i \in I$, there is a unique $\gamma_i^{f^{-1}} \in H_{\alpha(i)}$ such that $\gamma_i^{f^{-1}} \circ E(\hat{f}; \hat{f}^{-1})_i = (e^{\sigma_i})^{-1}$ holds. The family $\left\{ \gamma_i^{f^{-1}} \middle| i \in I \right\}$ induces a lift of the identity $\hat{\varepsilon}$ by Proposition E.3.3. We obtain another representative $\hat{\varepsilon} \circ \widetilde{\exp_{\mathrm{Orb}}} \circ \tau'$ of $E([\hat{\tau}^g])$, whose lifts $E_{\hat{f}}(\hat{g})_i := \gamma_i^{f^{-1}} \circ E(\hat{f}; \hat{g})_i$, $i \in I$ are smooth embeddings. Futhermore for $[\hat{g}] = [\hat{f}]^{-1}$, by construction assertion (b) holds. □

6.4.8 Remark (a) The construction of $E(\hat{f}; \hat{g})$ in Lemma 6.4.7 (combine $H_{\alpha(i)}.\,\mathrm{Im}\, e^{\sigma_i} \subseteq \Omega_{2,i}$ with Lemma 6.1.2 iii.) shows, that there are well-defined maps $E^{\hat{g}}_{\hat{f}} := \exp_{W_{\alpha(i)}} \circ \tau^g_{\alpha(i)}|_{H_{\alpha(i)}.(\mathrm{Im}\, e^{\sigma_i})}$ with $E^{\hat{g}}_{\hat{f}}|_{\mathrm{Im}\, e^{\sigma_i}} = E(\hat{f}; \hat{f}^{-1})$. As each $\tau_{\alpha(i)}$ is a canonical lift of an orbisection, from Step 3 in the proof of Lemma 6.1.2 we deduce $\eta \circ E^{\hat{g}}_{\hat{f}} = E^{\hat{g}}_{\hat{f}} \circ \eta$ for each $\eta \in H_{\alpha(i)}$.

(b) Let $[\hat{f}] = \mathrm{id}_{(Q,\mathcal{U})}$ and consider γ_i^f as in the proof of Lemma 6.4.7, then $\gamma_i^{\mathrm{id}_{(Q,\mathcal{U})}} = \mathrm{id}_{U_i}$ holds for each $i \in I$. To see this observe the identities $\mathrm{id}_{(Q,\mathcal{U})} = \mathrm{id}_{(Q,\mathcal{U})}^{-1}$ and $E^{-1}(\mathrm{id}_{(Q,\mathcal{U})}) = \mathbf{0}_{\mathrm{Orb}}$. For $i \in I$, both lifts constructed in (6.4.10) coincide as $\mathrm{id}_{U_i} = \exp_{W_{\alpha(i)}} \circ 0_{U_i}$. Thus forcing the identity $\gamma_i^{\mathrm{id}_{(Q,\mathcal{U})}} = \mathrm{id}_{U_i}$.

6.4.9 Definition For $[\hat{\phi}]$ in $\mathcal{S}$ let $[\widehat{\sigma^\phi}]$ be the unique orbisection with $E([\widehat{\sigma^\phi}]) = [\hat{\phi}]$. Apply Lemma 6.4.7 to $[\hat{\phi}]^{-1} \in \mathcal{S}$. By Part (b) of Lemma 6.4.7 we obtain a representative $\hat{\phi}$ of $[\hat{\phi}]$. For each $i \in I$ the lifts $g_i^\phi := E_{\hat{\phi}^{-1}}(\hat{\phi})_i$ of $\hat{\phi}$ are embeddings of $U_{\phi_i} := \exp_{W_{\alpha(i)}}(\sigma_i^{\phi^{-1}}(U_i)) \subseteq \Omega_{2,i}$ with $\operatorname{Im} g_i^\phi = U_i$. The pointwise operations turn

$$C_i^\phi := \left\{ f \in C^\infty(U_{\phi_i}, TW_{\alpha(i)}) \middle| \pi_{TW_{\alpha(i)}} \circ f = g_i^\phi \right\}$$

into a vector space. Endow this vector space with the unique topology turning the pullback $(g_i^\phi)^* \colon \mathfrak{X}(U_i) \to C_i^\phi, \sigma_i \mapsto \sigma_i \circ g_i^\phi$ into a linear topological isomorphism. We define a mapping

$$\Lambda_{[\hat{\phi}]} \colon C_{[\hat{\phi}]} := \left\{ [\hat{\sigma}] \circ [\hat{\phi}] \middle| [\hat{\sigma}] \in \mathfrak{X}_{\mathrm{Orb}}(Q)_c \right\} \to \bigoplus_{i \in I} C_i^\phi, [\hat{\sigma}] \circ [\hat{\phi}] \mapsto (\sigma_i \circ g_i^\phi)_{i \in I},$$

where σ_i is the canonical lift of $[\hat{\sigma}]$ on U_i. As orbisections are uniquely determined by a family of canonical lifts and a diffeomorphism of orbifolds is uniquely determined by any set of lifts, whose domains form an orbifold atlas, the map $\Lambda_{[\hat{\phi}]}$ is injective. Endow $C_{[\hat{\phi}]}$ with the unique locally convex topology turning $\Lambda_{[\hat{\phi}]}$ into a topological embedding.
The lifts $g_i^{\mathrm{id}_{(Q,\mathcal{U})}}$ are just the identity on U_i for each $i \in I$ by Remark 6.4.8 (b). Therefore $C_{\mathrm{id}_{(Q,\mathcal{U})}}$ and $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$ coincide, whence the mappings $\Lambda_{\mathrm{id}_{(Q,\mathcal{U})}}$ and $\Lambda_{\mathcal{A}}$ are the same.

For the rest of this section fix the notation of Definition 6.4.9. We obtain a structural result for the tangent manifold of $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$:

6.4.10 Lemma *Let $[\hat{\phi}]$ be an element of $\mathcal{S}$ with $\mathcal{S}$ as in Lemma 6.4.6. There is an isomorphism of topological vector spaces*

$$\alpha_{[\hat{\phi}]} \colon T_{[\hat{\phi}]}\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U}) \to \operatorname{Im} \Lambda_{[\hat{\phi}]},$$

whence $T_{[\hat{\phi}]}\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ is isomorphic as a topological vector space to $C_{[\hat{\phi}]}$.

Proof. Fix $[\hat{\phi}] \in \mathcal{S}$. As $\mathcal{S}$ is a symmetric set (i.e. $\mathcal{S} = \mathcal{S}^{-1}$), the inverse $[\hat{\phi}]^{-1}$ of $[\hat{\phi}]$ is contained in $\mathcal{S}$. By construction of $\mathcal{S}$, there is a representative of $[\hat{\phi}]^{-1}$ with lifts $\left\{ (g_i^\phi)^{-1} \colon U_i \to W_{\alpha(i)} \right\}_{i \in I}$. To shorten our notation, we set $U_{\phi_i} := (g_i^\phi)^{-1}(U_i)$ and recall $U_{\phi_i} \subseteq \Omega_{2,i}$ from Definition 6.4.9. The family of lifts $\left\{ g_i^\phi \middle| i \in I \right\}$ uniquely determines a representative of $[\hat{\phi}]$ by Corollary 3.1.12. We proceed in several steps:

Step 1: *Construct the mapping* $\alpha_{[\hat{\phi}]}$. For each $[\hat{g}] \in \mathcal{S}$ denote by $[\hat{\sigma}^g]$ the compactly supported orbisection with $E([\hat{\sigma}^g]) = [\hat{g}]$. By Lemma 6.4.7 (a) each $[\hat{g}] \in \mathcal{S}$ posseses a representative $E_{\hat{\phi}^{-1}}(\hat{g})$, with lifts $(E_{\hat{\phi}^{-1}}(\hat{g}))_i := \gamma_i^\phi \exp_{W_{\alpha(i)}} \circ \sigma^g_{\alpha(i)}|_{U_{\phi_i}}$. Fix $i \in I, p \in U_{\phi_i}$ and consider the map

$$\varepsilon_p^{\phi_i} \colon \mathcal{S} \to W_{\alpha(i)}, [\hat{g}] \mapsto E_{\hat{\phi}^{-1}}(\hat{g})_i(p).$$

We claim that $\varepsilon_p^{\phi_i}$ is a smooth map. To prove the claim let $\tau_{W_{\alpha(i)}} \colon \mathfrak{X}_{\mathrm{Orb}}(Q)_c \to \mathfrak{X}\left(W_{\alpha(i)}\right)$ be the map, which sends an orbisection to its canonical lift on $W_{\alpha(i)}$. By Definition 4.3.3 (b) this map is continuous linear, whence smooth. Choose a manifold chart (V_p, ψ_p) of the manifold $W_{\alpha(i)}$ with $p \in V_p$. The map $\tau_{V_p} \colon \mathfrak{X}\left(W_{\alpha(i)}\right) \to C^\infty(V_p, \mathbb{R}^d), X \mapsto X_{\psi_p} := \mathrm{pr}_2 T\psi_p X|_{V_p}$ is continuous linear by Definition C.3.1. Let $\varepsilon_p \colon C^\infty(V_p, \mathbb{R}^d) \to \mathbb{R}^d, f \mapsto f(p)$ be the evaluation map in p. This map is a linear map, which is smooth by [2, Proposition 3.20]. Finally we define the evaluation map $\mathrm{ev}_p \colon \mathfrak{X}\left(W_{\alpha(i)}\right) \to T_p W_{\alpha(i)}, X \mapsto X(p)$. As $\mathrm{ev}_p = (T_p\psi_p)^{-1} \circ \varepsilon_p \circ \tau_{V_p}$ holds, ev_p is continuous linear. By construction of $\mathcal{H}_\rho$, it is contained in the open subset $\mathcal{M}$ constructed in Proposition 6.1.5 (cf. Construction 6.1.6). Hence Lemma 6.1.2 ii. implies that ev_p maps $\tau_{W_{\alpha(i)}} \circ E^{-1}(\mathcal{S}) \subseteq \mathcal{H}_\rho$ to the set $\hat{O}_{\alpha(i)} \cap T_p W_{\alpha(i)}$. The image of the smooth map $\mathrm{ev}_p \circ \tau_{W_{\alpha(i)}} \circ E^{-1}|_{\mathcal{S}}$ is thus contained in $\operatorname{dom} \exp_{W_{\alpha(i)}} \cap T_p W_{\alpha(i)}$. By construction of the lifts $E_{\hat{\phi}^{-1}}(\hat{g})_i$ in Lemma 6.4.7, one may rewrite $\varepsilon_p^{\phi_i}$ as composition of smooth maps, thus proving the claim:

$$\varepsilon_p^{\phi_i} = \gamma_i^\phi \circ \exp_{W_{\alpha(i)}}|_{T_p W_{\alpha(i)}} \circ \mathrm{ev}_p \circ \tau_{W_{\alpha(i)}} \circ E^{-1}|_{\mathcal{S}}$$

Repeating the construction for each pair $p \in U_{\phi_i}$, where i runs through I, we obtain a map

$$\begin{aligned} \alpha_{[\hat{\phi}]} \colon T_{[\hat{\phi}]} \operatorname{Diff}_{\mathrm{Orb}}(Q,\mathcal{U}) &\to \prod_{i \in I} (TW_{\alpha(i)})^{U_{\phi_i}} \\ V &\mapsto (T_{[\hat{\phi}]} \varepsilon_p^{\phi_i}(V))_{i \in I, p \in U_{\phi_i}} \end{aligned}$$

and abbreviate its image as $V_{[\hat{\phi}]} := \operatorname{Im} \alpha_{[\hat{\phi}]}$.

Step 2: *Endow* $V_{[\hat{\phi}]}$ *with a vector space strucure which turns* $\alpha_{[\hat{\phi}]}$ *into a linear map.*
The tangent space $T_{[\hat{\phi}]} \operatorname{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ is the set of equivalence classes of C^1-curves $\eta \colon]-\varepsilon, \varepsilon[\to \mathcal{S}$ with $\eta(0) = [\hat{\phi}]$, where $\eta \sim \theta$ if and only if $(E^{-1} \circ \eta)'(0) = (E^{-1} \circ \theta)'(0)$ holds. Abbreviate the equvialence classes with respect to this relation by $[t \mapsto \eta(t)]_\sim$. Since each $\varepsilon_p^{\phi_i}$ is smooth and η is of class C^1, for each $i \in I, p \in U_{\phi_i}$ the curve $\varepsilon_p^{\phi_i} \circ \eta$ is of class C^1. Hence the definition of $\alpha_{[\hat{\phi}]}$ yields

$$\alpha_{[\hat{\phi}]}([\eta]_\sim) = ([t \mapsto E_{[\hat{\phi}]^{-1}}(\eta(t))_i(p)]_\sim)_{i \in I, p \in U_{\phi_i}}. \tag{6.4.11}$$

The curve η in (6.4.11) passes through $[\hat{\phi}]$ for $t = 0$, whence by Lemma 6.4.7 (b) for $i \in I$, $E_{\hat{\phi}^{-1}}(\gamma(0))_i = g_i^\phi$ holds. Therefore we infer from (6.4.11) the identity

$$V_{[\hat{\phi}]} \subseteq \left\{ (f_i)_{i \in I} \in \prod_{i \in I} (TW_{\alpha(i)})^{U_{\phi_i}} \middle| \forall i \in I, p \in U_{\phi_i},\ f_i(p) \in T_{g_i^\phi(p)} W_{\alpha(i)} \right\}. \tag{6.4.12}$$

In particular (6.4.12) shows that the pointwise operations turn $V_{[\hat{\phi}]}$ into a vector space. Furthermore by (6.4.12) $T_{[\hat{\phi}]} \varepsilon_p^{\phi_i} \colon T_{[\hat{\phi}]} \operatorname{Diff}_{\mathrm{Orb}}(Q,\mathcal{U}) \to T_{g_i^\phi(p)} W_{\alpha(i)}$ is linear. By definition the map $\alpha_{[\hat{\phi}]}$ becomes linear, if $V_{[\hat{\phi}]}$ is endowed with the vector space structure induced by pointwise operations.

Step 3: *A formula relating $\alpha_{[\hat{\phi}]}$ to $\alpha_{\mathrm{id}_{(Q,\mathcal{U})}}$.* Let $\rho_{[\hat{\phi}]}\colon \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U}) \to \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U}), [\hat{\psi}] \mapsto [\hat{\psi}]\circ[\hat{\phi}]$ be the right translation and define

$$G^\phi := ((g_i^\phi)_{i\in I})^*\colon \prod_{i\in I}(TW_{\alpha(i)})^{U_i} \to \prod_{i\in I}(TW_{\alpha(i)})^{U_{\phi_i}}, (f_i)_{i\in I} \mapsto (f_i \circ g_i^\phi)_{i\in I}$$

Consider $[\eta]_\sim \in T_{\mathrm{id}_{(Q,\mathcal{U})}}\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$. The composition in $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ is continuous, as it is a Lie group. Since $\eta(0) = \mathrm{id}_{(Q,\mathcal{U})}$ holds, we may thus assume $[\gamma(t)]\circ[\hat{\phi}] \in \mathcal{S}$ for all t. By Lemma 6.4.7 (a), there is a representative of $\gamma(t)\circ[\hat{\phi}]$ with lifts $E_{\hat{\phi}^{-1}}(\eta(t)\circ\hat{\phi})_i = \gamma_i^\phi \exp_{W_{\alpha(i)}} \sigma_{\alpha(i)}^{\eta(t)\circ\phi}|_{U_{\phi_i}}$. Here $\sigma_{\alpha(i)}^{\eta(t)\circ\phi}$ is the canonical lift on $W_{\alpha(i)}$ of the compactly supported orbisection $[\widehat{\sigma^{\gamma(t)\circ\phi}}]$ with $E([\widehat{\sigma^{\gamma(t)\circ\phi}}]) = \gamma(t)\circ[\hat{\phi}]$. The set U_{ϕ_i} is contained in $\Omega_{2,i} \subseteq \Omega_{\frac{5}{4},K_{5,i}}$ (cf. Construction 6.1.6). Uniqueness of canonical lifts and (6.1.4) in Lemma 6.1.7 imply $\sigma_{\alpha(i)}^{\eta(t)\circ\phi}|_{U_{\phi_i}} = \sigma_{\alpha(i)}^{\eta(t)} \diamond_i \sigma_{\alpha(i)}^{\phi}|_{U_{\phi_i}}$. Recall that by construction of $\sigma_{\alpha(i)}^{\eta(t)} \diamond_i \sigma_{\alpha(i)}^{\phi}|_{U_{\phi_i}}$ (see (D.0.17) in Construction D.0.8) the identity

$$\exp_{W_{\alpha(i)}} \circ \sigma_{\alpha(i)}^{\eta(t)} \diamond_i \sigma_{\alpha(i)}^{\phi}|_{U_{\phi_i}} = \exp_{W_{\alpha(i)}} \circ\sigma_{\alpha(i)}^{\eta(t)} \circ \exp_{W_{\alpha(i)}} \circ\sigma_{\alpha(i)}^{\phi}|_{U_{\phi_i}}$$

holds. Furthermore $g_i^\phi = E_{\hat{\phi}^{-1}}(\hat{\phi})_i = \gamma_i^\phi \circ \exp_{W_{\alpha(i)}} \circ\sigma_{\alpha(i)}^{\phi}|_{U_{\phi_i}}$ and $\operatorname{Im} g_i^\phi = U_i$ hold. Hence $\exp_{W_{\alpha(i)}} \circ\sigma_{\alpha(i)}^{\phi}(U_{\phi_i}) \subseteq H_{\alpha(i)}.U_i \subseteq \Omega_{2,i}$ follows. Analogous to Step 2 in the proof of Lemma 6.1.2 one shows that $\gamma_i^\phi \in H_{\alpha(i)}$ commutes with $\exp_{W_{\alpha(i)}} \circ\sigma_{\alpha(i)}^{\eta(t)}|_{H_{\alpha(i)}.U_i}$. Summing up we obtain:

$$\begin{aligned}
\alpha_{[\hat{\phi}]}(T\rho_{[\hat{\phi}]}([\eta]_\sim)) &= \Big([t \mapsto E_{\hat{\phi}^{-1}}(\eta(t)\circ\hat{\phi})_i(p)]_\sim\Big)_{i\in I, p\in U_{\phi_i}} \\
&= \Big([t \mapsto \gamma_i^\phi \circ \exp_{W_{\alpha(i)}} \sigma_{\alpha(i)}^{\eta(t)} \diamond_i \sigma_{\alpha(i)}^{\phi}(p)]_\sim\Big)_{i\in I, p\in U_{\phi_i}} \\
&= \Big([t \mapsto \gamma_i^\phi \exp_{W_{\alpha(i)}} \sigma^{\eta(t)} \exp_{W_{\alpha(i)}} \sigma_{\alpha(i)}^{\phi}(p)]_\sim\Big)_{i\in I, p\in U_{\phi_i}} \\
&= \left([t \mapsto \exp_{W_{\alpha(i)}} \sigma^{\eta(t)} \underbrace{\gamma_i^\phi \exp_{W_{\alpha(i)}} \sigma_{\alpha(i)}^{\phi}}_{=g_i^\phi}(p)]_\sim\right)_{i\in I, p\in U_{\phi_i}} \\
&= G^\phi \circ \alpha_{\mathrm{id}_{(Q,\mathcal{U})}}([\eta]_\sim)
\end{aligned}$$

Hence we derive $\alpha_{[\hat{\phi}]} \circ T\rho_{[\hat{\phi}]}|_{\mathrm{dom}\,\alpha_{\mathrm{id}_{(Q,\mathcal{U})}}} = G^\phi \circ \alpha_{\mathrm{id}_{(Q,\mathcal{U})}}$. Thus $G^\phi(V_{\mathrm{id}_{(Q,\mathcal{U})}}) = V_{[\hat{\phi}]}$ follows, as $T\rho_{[\hat{\phi}]}$ is a diffeomorphism.

Step 4: *$G^\phi|_{V_{\mathrm{id}_{(Q,\mathcal{U})}}}$ is linear.* To see this, let $v, w \in T_{\mathrm{id}_{(Q,\mathcal{U})}}\mathrm{Diff}_{\mathrm{Orb}}(Q)$ and $r \in \mathbb{R}$. Since $T\rho_{[\hat{\phi}]}$ and $\alpha_{\mathrm{id}_{(Q,\mathcal{U})}}$ are linear, the formula in Step 3 yields:

$$\begin{aligned}
G^\phi(\alpha_{\mathrm{id}_{(Q,\mathcal{U})}}(v) + r\alpha_{\mathrm{id}_{(Q,\mathcal{U})}}(w)) &= G^\phi(\alpha_{\mathrm{id}_{(Q,\mathcal{U})}}(v + rw)) = \alpha_{[\hat{\phi}]}(T\rho_{[\hat{\phi}]}(v+rw)) \\
&= \alpha_{[\hat{\phi}]}(T\rho_{[\hat{\phi}]}(v)) + r\alpha_{[\hat{\phi}]}(T\rho_{[\hat{\phi}]}(w)) \\
&= G^\phi(\alpha_{\mathrm{id}_{(Q,\mathcal{U})}}(v)) + rG^\phi(\alpha_{\mathrm{id}_{(Q,\mathcal{U})}}(w)),
\end{aligned}$$

Step 5: $\alpha_{\mathrm{id}_{(Q,\mathcal{U})}}$ *is an isomorphism of topological vector spaces and* $V_{\mathrm{id}_{(Q,\mathcal{U})}} = \mathrm{Im}\,\Lambda_{\mathcal{A}}$ *holds.* Consider the map $h\colon \mathfrak{X}_{\mathrm{Orb}}(Q)_c \to T_{\mathrm{id}_{(Q,\mathcal{U})}}\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U}), [\hat{\sigma}] \mapsto [t \mapsto E(t[\hat{\sigma}])]$. For $i \in I$ we denote by σ_i the canonical lift on U_i of the orbisection $[\hat{\sigma}]$. Then (6.4.11) together with Remark 6.4.8 (b) and (6.4.10) imply:

$$\alpha_{\mathrm{id}_{(Q,\mathcal{U})}} \circ h([\hat{\sigma}]) = ([t \mapsto \exp_{W_{\alpha(i)}}(t\sigma_i(p))])_{i\in I, p\in U_i} \tag{6.4.13}$$

As $\exp_{W_{\alpha(i)}}$ is the Riemannian exponential map on $W_{\alpha(i)}$, the map $c_{i,p}(t) := \exp_{W_{\alpha(i)}}(t\sigma_i(p))$ is a geodesic with $c'_{i,p}(0) = \sigma_i(p)$. Therefore (6.4.13) yields $\alpha_{\mathrm{id}_{(Q,\mathcal{U})}} \circ h([\hat{\sigma}]) = (\sigma_i)_{i\in I} = \Lambda_{\mathcal{A}}([\hat{\sigma}])$. It is well known that h is an isomorphism of topological vector spaces (see [50, Definition I.3.3.]). Hence $V_{\mathrm{id}_{(Q,\mathcal{U})}} = \mathrm{Im}\,\alpha_{\mathrm{id}_{(Q,\mathcal{U})}} \subseteq \mathrm{Im}\,\Lambda_{\mathcal{A}} \subseteq \bigoplus_{i\in I}\mathfrak{X}(U_i)$ holds and $\alpha_{\mathrm{id}_{(Q,\mathcal{U})}} = \Lambda_{\mathcal{A}} \circ h^{-1}$ implies $V_{\mathrm{id}_{(Q,\mathcal{U})}} = \mathrm{Im}\,\Lambda_{\mathcal{A}}$ and $\alpha_{\mathrm{id}_{(Q,\mathcal{U})}}$ is an isomorphism of topological vector spaces. In particular the formula shows that $\alpha_{\mathrm{id}_{(Q,\mathcal{U})}}$ is a linear isomorphism onto the closed subspace $V_{\mathrm{id}_{(Q,\mathcal{U})}} = \mathrm{Im}\,\Lambda_{\mathcal{A}} \subseteq \bigoplus_{i\in I}\mathfrak{X}(U_i)$.

Step 6: $G^{\phi}|_{V_{\mathrm{id}_{(Q,\mathcal{U})}}}^{V_{[\hat{\phi}]}}$ *is an isomorphism of topological vector spaces.* By definition G^{ϕ} is the map $(g_i^{\phi})^*_{i\in I}$ and each $g_i^{\phi}\colon U_{\phi_i} \to U_i$ is a diffeomorphism. The map $(g_i^{\phi})^*\colon \mathfrak{X}(U_i) \to C_i^{\phi}$ is an isomorphism of topological vector spaces by Definition 6.4.9. From [10, Proposition II.31 8 (i)] we deduce that the mapping $G^{\phi}|_{\bigoplus_{i\in I}\mathfrak{X}(U_i)}^{\bigoplus_{i\in I}C_i^{\phi}}$ is an isomorphism of topological vector spaces. By Step 5, $V_{\mathrm{id}_{(Q,\mathcal{U})}}$ is a subspace of $\bigoplus_{i\in I}\mathfrak{X}(U_i)$ and $V_{[\hat{\phi}]} = G^{\phi}(V_{\mathrm{id}_{(Q,\mathcal{U})}})$ holds by Step 3. Since G^{ϕ} maps $\bigoplus_{i\in I}\mathfrak{X}(U_i)$ into $\bigoplus_{i\in I}C_{\phi_i}$, the set $V_{[\hat{\phi}]}$ is contained in $\bigoplus_{i\in I}C_{\phi_i}$Endowing $V_{\mathrm{id}_{(Q,\mathcal{U})}}$ with the subspace topology of $\bigoplus_{i\in I}\mathfrak{X}(U_i)$ and $V_{[\hat{\phi}]}$ with the subspace topology of $\bigoplus_{i\in I}C_i^{\phi}$, the map $G^{\phi}|_{V_{\mathrm{id}_{(Q,\mathcal{U})}}}^{V_{[\hat{\phi}]}}$ becomes an isomorphism of topological vector spaces. By construction for $(f_i)_{i\in I} \in V_{[\hat{\phi}]}$ there is a unique $[\widehat{\sigma^f}] \in \mathfrak{X}_{\mathrm{Orb}}(Q)_c$ such that $(f_i)_{i\in I} = G^{\phi}\Lambda_{\mathcal{A}}([\widehat{\sigma^f}]) = (\sigma_i^f \circ g_i^{\phi})_{i\in I}$ holds. Hence the elements in $V_{[\hat{\phi}]}$ are of the form $(\sigma_i \circ g_i^{\phi})_{i\in I}$, where σ_i is the canonical representative on U_i of some $[\hat{\sigma}] \in \mathfrak{X}_{\mathrm{Orb}}(Q)_c$. As a consequence of the definition of $\Lambda_{[\hat{\phi}]}$, as a set $\mathrm{Im}\,\Lambda_{[\hat{\phi}]}$ and $V_{[\hat{\phi}]}$ coincide. By definition of the topology they also coincide as topological vector spaces.

Step 7: $\alpha_{[\hat{\phi}]}$ *is an isomorphism of topological spaces for each* $[\hat{\phi}] \in \mathcal{S}$. Endowing $V_{[\hat{\phi}]}$ with the topology as in Step 6, we obtain a commutative diagramm for $[\hat{\phi}] \in \mathcal{S}$:

$$\begin{tikzcd}
T_{\mathrm{id}_{(Q,\mathcal{U})}}\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U}) \arrow[r, "\alpha_{\mathrm{id}_{(Q,\mathcal{U})}}"] \arrow[d, "T\rho_{[\hat{\phi}]}"] & V_{\mathrm{id}_{(Q,\mathcal{U})}} \arrow[d, "G^{\phi}|_{V_{\mathrm{id}_{(Q,\mathcal{U})}}}^{V_{[\hat{\phi}]}}"] \\
T_{[\hat{\phi}]}\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U}) \arrow[r, "\alpha_{[\hat{\phi}]}"] & V_{[\hat{\phi}]}
\end{tikzcd}$$

As all arrows with the exception of the lower row are isomorphisms of topological vector spaces, so is $\alpha_{[\hat{\phi}]}$. By Step 6, $\mathrm{Im}\,\alpha_{[\hat{\phi}]} = V_{[\hat{\phi}]} = \mathrm{Im}\,\Lambda_{[\hat{\phi}]}$ holds, thus proving the assertion. $\square$

We are now in a position to obtain regularity properties for the Lie group $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$.

6.4.11 Theorem *Let $(Q,\mathcal{U})$ be σ-compact. The Lie group $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ is C^k-regular for each $k \in \mathbb{N}_0 \cup \{\infty\}$. In particular this group is regular in the sense of Milnor.*

Proof. We claim that $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ is a (strong) C^0-regular Lie group. If this were true, the assertion is a direct consequence of Definition C.5.3. To prove the claim, by Lemma C.5.4 it suffices to obtain a smooth evolution and right product integrals for any zero-neighborhood $C^0([0,1],U)$. Let $E\colon \mathcal{H}_\rho \to \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U}), [\hat{\sigma}] \to [\exp_{\mathrm{Orb}}] \circ [\hat{\sigma}]|^{\Omega}$ be the manifold chart at the identity introduced in Theorem 6.2.4 (cf. Proposition 6.1.5). Using the map evol introduced in Lemma 6.4.5, we define a map

$$E_1 := E \circ \mathrm{evol}|_{C^0([0,1],\mathcal{R})}\colon C^0([0,1],\mathcal{R}) \to \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U}),$$

where $\mathcal{R}$ is chosen as in Lemma 6.4.6 with respect to the symmetric subset $\mathcal{S} \subseteq \mathrm{Im}\, E$. By Lemma 6.4.5 evol is a smooth map, whence E_1 is smooth as a composition of smooth maps. Following Lemma C.5.4, the Lie group $\mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ will be (strong) C^0-regular if we can show that each $\gamma \in C^0([0,1],\mathcal{R})$ has a right product integral $\mathcal{P}(\gamma)$ with $\mathcal{P}(\gamma)(1) = E_1(\gamma)$.

To this end, consider a C^1-curve $\eta\colon [0,1] \to \mathcal{S} \subseteq \mathrm{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$. For $s \in [0,1]$ we let $[\hat{\sigma}^{\eta(s)}]$ be the preimage $E^{-1}(\eta(s))$ (cf. Defintion 6.4.9). Recall from Lemma 6.4.7 that for each $s,t \in [0,1]$ there is a representativ $E_{\eta(t)^{-1}}(\eta(s))$ of $\eta(s)$. Using the notation of Defintion 6.4.9, the lifts of this representative with respect to the atlas $\left\{ (U_{\eta(t)_i}, H_{\alpha(i),U_{\eta(t)_i}}, \varphi_{\alpha(i)}|_{U_{\eta(t)_i}}) \right\}_{i\in I}$ are given as

$$E_{\eta(t)^{-1}}(\eta(s))_i = \gamma_i^{\eta(t)}.\exp_{W_{\alpha(i)}} \circ \sigma_{\alpha(i)}^{\eta(s)}|_{U_{\eta(t)_i}}$$

The derivative of the lift with respect to s may be computed locally in manifold-charts. To do so, we fix $(s,p) \in [0,1] \times U_{\eta(t)_i}$ for some $t \in [0,1]$: Since $U_{\eta(t)_i} \subseteq \Omega_{2,i}$ holds by Definition 6.4.9, we choose and fix a manifold chart $(V^{n_p}_{5,\alpha(i)}, \kappa^{\alpha(i)}_{n_p}) \in \mathcal{F}_5(K_{5,i})$ with $p \in V^{n_p}_{1,\alpha(i)}$. Observe that by [25, Lemma F.6 and Lemma 4.11] the map $K^{\alpha(i)}_{n_p}\colon \mathfrak{X}\left(V^{n_p}_{5,\alpha(i)}\right) \to C^\infty(B_5(0),\mathbb{R}^d), X \mapsto X_{[n_p]}$, with $X_{[n_p]} = C^\infty((\kappa^{\alpha(i)}_{n_p})^{-1},\mathbb{R}^d)(\theta_{\kappa^{\alpha(i)}_n}(X))$ is an isomorphism of topological vector spaces. As η is of class C^1, the following composition yields a C^1-curve,

$$\eta_{t,p,i} := K^{\alpha(i)}_{n_p} \circ \mathrm{res}^{W_{\alpha(i)}}_{V^{n_p}_{5,\alpha(i)}} \circ \tau_{W_{\alpha(i)}} \circ E^{-1} \circ \eta\colon [0,1] \to C^\infty(B_5(0),\mathbb{R}^d).$$

Let $\exp_{n_p}$ be the Riemannian exponential map induced on $B_5(0)$ by the pullback metric of the Riemannian metric on $W_{\alpha(i)}$ via $\kappa^{\alpha(i)}_{n_p}$. Since $E^{-1}(\mathcal{S}) \subseteq \mathcal{H}_\rho$ and $(V^{n_p}_{5,\alpha(i)}, \kappa^{\alpha(i)}_{n_p}) \in \mathcal{F}_{5,K_{5,i}}$, the construction of $\mathcal{H}_\rho$ (cf. Theorem 6.2.4 or more precicsely Construction 6.1.6 and Construction D.0.8) shows $\eta_{t,p,i}([0,1])(\overline{B_3(0)}) \subseteq B_{\varepsilon_{n_p}}(0) \subseteq B_{\nu_{n_p}}(0)$, whence $\eta_{t,p,i}(s) \in \lfloor \overline{B_2(0)}, B_{\nu_{n_p}}(0) \rfloor_\infty \subseteq C^\infty(B_5(0),\mathbb{R}^d)$ holds for all $s \in [0,1]$. By choice of ν_{n_p}, the set $B_4(0) \times B_{\nu_{n_p}}(0)$ is contained in $\mathrm{dom}\exp_{n_p}$ (cf. Lemma D.0.6). We deduce from [25, Proposition 4.23] that

$$(\exp_{n_p})_*\colon \lfloor \overline{B_2(0)}, \mathrm{dom}\, B_{\nu_{n_p}}(0) \rfloor_\infty \to C^\infty(B_2(0),\mathbb{R}^d), f \mapsto \exp_{n_p}(\mathrm{id}_{B_2(0)}, f|_{B_2(0)})$$

is smooth and on $B_2(0) \times B_{\nu_{n_p}}(0)$. We obtain a C^1-curve $(\exp_{n_p})_* \circ \eta_{t,p,i}\colon [0,1] \to C^\infty(B_2(0),\mathbb{R}^d)$. Furthermore Lemma D.0.6 (b) yields $\exp_{W_{\alpha(i)}} \circ T\kappa^{\alpha(i)}_{n_p}|_{B_2(0)\times B_{\nu_{n_p}}(0)} = \kappa^{\alpha(i)}_{n_p} \circ \exp_{n_p}|_{B_2(0)\times B_{\nu_{n_p}}(0)}$.

The above considerations did not depend on $p \in U_{\eta(t)_i}$, whence they may be repeated for each $p \in U_{\eta(t)_i}, i \in I$. With Lemma D.0.6 (b) and and the Exponential law [2, Theorem 3.28] we may now compute the derivative as:

$$\begin{aligned}\alpha_{\eta(t)}(\eta'(t+s)) &= \alpha_{\eta(t)}([s \mapsto \eta(t+s)]_\sim) = ([s \mapsto E_{(\eta(t))^{-1}}(\eta(t+s))_i(s)(p)]_\sim)_{i\in I, p\in U_{\gamma(t)_i}}\\ &= ([s \mapsto \gamma_i^{\eta(t)} \exp_{W_{\alpha(i)}} \sigma_{\alpha(i)}^{\eta(t+s)}(p)]_\sim)_{i\in I,p\in U_{\gamma(t)_i}}\\ &= ([s\mapsto \gamma_i^{\eta(t)} \exp_{W_{\alpha(i)}} T\kappa_{n_p}^{\alpha(i)}(T\kappa_{n_p}^{\alpha(i)})^{-1}(\kappa_{n_p}^{\alpha(i)}(p), \eta_{t,p,i}(t+s)(\kappa_{n_p}^{\alpha(i)}(p))])_{i\in I,p\in U_{\gamma(t)_i}}\\ &= ([s\mapsto \gamma_i^{\eta(t)}\kappa_{n_p}^{\alpha(i)}(\exp_{n_p})_* \circ \eta_{t,p,i}(t+s)(\kappa_{n_p}^{\alpha(i)}(p))]_\sim)_{i\in I,p\in U_{\gamma(t)_i}}\\ &= (T(\gamma_i^{\eta(t)}\kappa_{n_p}^{\alpha(i)})d_1((\exp_{n_p})_*\circ\eta_{t,p,i})^\wedge(t+s,\kappa_{n_p}^{\alpha(i)}(p)))_{i\in I,p\in U_{\gamma(t)_i}} \end{aligned} \tag{6.4.14}$$

Let $\xi \in C^0([0,1],\mathcal{R})$ be some continuous curve. By Lemma 6.4.6 we may consider the C^1-curve $\eta := E\circ\omega(\eta)\colon [0,1]\to\mathcal{S}$. To compute the derivative $\eta'(t)$ we exploit the identity (6.4.14). The definition of the mappings implies

$$\eta_{t,p,i} = K_{n_p}^{\alpha(i)} \circ \mathrm{res}_{V_{5,\alpha(i)}^{n_p}}^{W_{\alpha(i)}} \tau_{W_{\alpha(i)}} \circ E^{-1}(E\circ\omega(\xi)) = K_{n_p}^{\alpha(i)}(\omega(\xi)_{\alpha(i)}|_{V_{5,\alpha(i)}^{n_p}}).$$

The canonical lift $\omega(\xi)_{\alpha(i)}$ is uniquely determined, whence $\omega(\xi)_{\alpha(i)}$ coincides with $\omega_i(\xi_{\alpha(i)})$ (cf. Lemma 6.4.3) on $\Omega_{2,K_{5,i}}$ by the proof of Lemma 6.4.4. Since $(V_{5,\alpha(i)}^{n_p}, \kappa_{n_p}^{\alpha(i)}) \in \mathcal{F}_5(K_{5,i})$ holds, we derive $V_{2,\alpha(i)}^{n_p} \subseteq \Omega_{2,K_{5,i}}$. Therefore the lift satisfies (6.4.4). Summing up for $(s,x)\in[0,1]\times V_{2,\alpha(i)}^{n_p}$:

$$\begin{aligned}\eta_{t,p,i}(s)(x) &= K_{n_p}^{\alpha(i)}(e(\xi)_{\alpha(i)})(t)(x)\\ &= \mathrm{pr}_2 \circ T\kappa_{n_p}^{\alpha(i)}(\exp_{W_{\alpha(i)}}|_{N_x})^{-1}\circ(\kappa_{n_p}^{\alpha(i)})^{-1}\circ \mathrm{Fl}_0^f(s,\kappa_{n_p}^{\alpha(i)}(x),\xi_{\alpha(i)[n_p]})\end{aligned}$$

Observe that $(\exp_{n_p})_* \,\mathrm{pr}_2\, T\kappa_{n_p}^{\alpha(i)}(\exp_{W_{\alpha(i)}}|_{N_x})^{-1} = \exp_{n_p} T\kappa_{n_p}^{\alpha(i)}(\exp_{W_{\alpha(i)}}|_{N_x})^{-1}$ holds. By construction of N_x (see Lemma D.0.6 (b)) we obtain:

$$(\exp_{n_p})_*\,\mathrm{pr}_2\, T\kappa_{n_p}^{\alpha(i)}(\exp_{W_{\alpha(i)}}|_{N_x})^{-1} = \kappa_{n_p}^{\alpha(i)}\exp_{W_{\alpha(i)}}\circ(\exp_{W_{\alpha(i)}}|_{N_x})^{-1} = \kappa_{n_p}^{\alpha(i)}.$$

Insert this identity and the local formula for $\eta_{t,p,i}$ into (6.4.14):

$$\begin{aligned}\alpha_{\eta(t)}(\eta'(t)) &= \Big(T(\gamma_i^{\eta(t)}(\kappa_{n_p}^{\alpha(i)})^{-1})d_1((\exp_{n_p})_*\circ\eta_{t,p,i})^\wedge(t,\kappa_{n_p}^{\alpha(i)}(p))\Big)_{i\in I,p\in U_{\eta(t)_i}}\\ &= \Big(T(\gamma_i^{\eta(t)}(\kappa_{n_p}^{\alpha(i)})^{-1})d_1(\kappa_{n_p}^{\alpha(i)}(\kappa_{n_p}^{\alpha(i)})^{-1}\,\mathrm{Fl}_0^f(t,\kappa_{n_p}^{\alpha(i)}(p),\xi_{\alpha(i)[n_p]}))\Big)_{i\in I,p\in U_{\eta(t)_i}}\\ &= \Big(T(\gamma_i^{\eta(t)}(\kappa_{n_p}^{\alpha(i)})^{-1})d_1\,\mathrm{Fl}_0^f(t,\kappa_{n_p}^{\alpha(i)}(p),\xi_{\alpha(i)[n_p]})\Big)_{i\in I,p\in U_{\eta(t)_i}}\end{aligned}$$

Fixing $\kappa_{n_p}^{\alpha(i)}(p)$ and ξ, the flow $\mathrm{Fl}_0^f(t,\kappa_{n_p}^{\alpha(i)}(p),\xi_{\alpha(i)[n_p]})$ is a solution to the differential equation (6.4.1). Using the local representative of the vector field, the equation yields

$$\begin{aligned}d_1\,\mathrm{Fl}_0^f(t,\kappa_{n_p}^{\alpha(i)}(p),\xi_{\alpha(i)[n_p]}) &= (\mathrm{Fl}_0^f(t,\kappa_{n_p}^{\alpha(i)}(p),\xi_{\alpha(i)[n_p]}),\xi_{\alpha(i)[n_p]}(t)(\mathrm{Fl}_0^f(t,\kappa_{n_p}^{\alpha(i)}(p),\xi_{\alpha(i)[n_p]})))\\ &= T\kappa_{n_p}^{\alpha(i)}\xi(t)_{\alpha(i)}\circ(\kappa_{n_p}^{\alpha(i)})^{-1}(\mathrm{Fl}_0^f(t,\kappa_{n_p}^{\alpha(i)}(p),\xi_{\alpha(i)[n_p]})).\end{aligned}$$

Since $\xi(t)_{\alpha(i)}$ is a canonical lift, it is equivariant with respect to $H_{\alpha(i)}$. Thus the last identity proves:

$$\begin{aligned}\alpha_{\eta(t)}(\eta'(t)) &= (T(\gamma_i^{\eta(t)})\xi_{\alpha(i)}(t)(\kappa_{n_p}^{\alpha(i)})^{-1} \operatorname{Fl}_0^f(t, \kappa_{n_p}^{\alpha(i)}(p), \xi_{\alpha(i)[n_p]}))_{i\in I, p\in U_{\eta(t)_i}} \\ &= (\xi_{\alpha(i)}(t)(\gamma_i^{\eta(t)}.(\kappa_{n_p}^{\alpha(i)})^{-1} \operatorname{Fl}_0^f(t, \kappa_{n_p}^{\alpha(i)}(p), \xi_{\alpha(i)[n_p]}))_{i\in I, p\in U_{\eta(t)_i}}\end{aligned}$$

Furthermore $\omega(\xi)(t) = E^{-1}(\eta(t))$ holds by construction. Using the notation of Lemma 6.4.7 we obtain $\exp_{W_{\alpha(i)}} \circ\omega(\xi)(t)_{\alpha(i)}(p) = E(\eta(t)^{-1}, \eta(t))_i(p)$. On the other hand (6.4.4) yields the identity

$$\exp_{W_{\alpha(i)}} \circ\omega(\xi)(t)_{\alpha(i)}(p) = (\kappa_{n_p}^{\alpha(i)})^{-1} \operatorname{Fl}_0^f(t, \kappa_{n_p}^{\alpha(i)}(p), \xi_{\alpha(i)[n_p]}).$$

By choice of $\gamma_i^{\eta(t)}$ (see the proof of Lemma 6.4.7) we derive:

$$\alpha_{\eta(t)}(\eta'(t)) = (\xi_{\alpha(i)}(t)(g_i^{\eta(t)}(p))_{i\in I, p\in U_{\eta(t)_i}} = (\xi_{\alpha(i)}(t)(g_i^{\eta(t)}))_{i\in I}$$

We may now use the structural results on the tangent space of $\operatorname{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ at $\gamma(t) \in \mathcal{S}$. From Lemma 6.4.10 and its proof (in particular the formula in Step 3), we infer

$$\Lambda_{\eta(t)}^{-1}(\alpha_{\eta(t)}(\eta'(t))) = \xi(t)\circ\eta(t) = \Lambda_{\eta(t)}^{-1}(G^{\eta(t)}\alpha_{\mathrm{id}_{(Q,\mathcal{U})}}(\xi(t))) = \Lambda_{\eta(t)}^{-1}(\alpha_{\eta(t)}(T\rho_{\eta(t)}(\xi(t))).$$

The mapping $\Lambda_{\eta(t)}^{-1}\circ\alpha_{\eta(t)}$ is an isomorphism of topological vector spaces, whence $\eta'(t) = T\rho_{\eta(t)}(\xi(t))$ follows. Recalling the definition of η we have $\eta'(t) = (E(\omega(\xi)(t)))' = T\rho_{E(\omega(\xi)(t))}(\xi(t))$.
The facts obtained so far allow the computation of the right logarithmic derivative of $\eta(t) = E(\omega(\eta)(t))$:

$$\delta^r(\eta)(t) = T\rho_{E(\omega(\xi)(t))^{-1}}(E(\omega(\xi)(t)))' = T\rho_{E(\omega(\xi)(t))^{-1}}T\rho_{E(\omega(\xi)(t))}(\xi(t)) = \xi(t) \tag{6.4.15}$$

By construction $E_1(\xi) = E(\omega(\xi)(1)) = \eta(1)$ holds and Lemma 6.4.5 implies $\omega(\xi)(0) = \mathbf{0}_{\mathrm{Orb}}$ hold. Thus $\eta(0) = E(\omega(\xi)(0)) = E(\mathbf{0}_{\mathrm{Orb}}) = \mathrm{id}_{(Q,\mathcal{U})}$ holds. Furthermore the computation of the right logarithmic derivative (6.4.15) shows that the curve ξ posseses a right product integral $E(\omega(\xi)) = \eta$. We have already seen that the mapping E_1 is smooth, thus the proof is complete and $\operatorname{Diff}_{\mathrm{Orb}}(Q,\mathcal{U})$ is a (strong) C^0-regular Lie-group. □

6.4.12 Remark In general the orbifolds in the present paper are not assumed to be second countable. We had to require second countability of the orbifold, to assure that $\mathfrak{X}_{\mathrm{Orb}}(Q)_c$ is countably patched. In this case we obtain an atlas indexed by the countable set I, whence the map

$$\Lambda\colon \bigoplus_{i\in I} C^r([0,1], \mathfrak{X}(U_i)) \to C^r([0,1], \bigoplus_{i\in I}\mathfrak{X}(U_i)), (f_i) \mapsto \sum_{i\in I}(\iota_i)_*(f_i)$$

is an isomorphism of topological vector spaces for $r \in \mathbb{N}_0$ (see Lemma C.3.5. This fact was crucial to prove the smoothness of the evolution map evol. It is known that this map fails to be an isomorphism if I is uncountable (a proof for this fact has been communicated to the author by S.A. Wegner). Hence our methods do not generalize to the setting of arbitrary paracompact orbifolds.

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A. Hyperplanes and paths in euclidean space

The results in this Appendix are folklore. For the readers convenience we provide full proofs for these well known facts. As usual a hyperplane H in euclidean space $\mathbb{R}^d$ will be a linear subspace of codimension 1 and a path is a continuous linear map from an intervall into $\mathbb{R}^d$.

A.1 Lemma *Let $d \in \mathbb{N}$ and $X \subseteq \mathbb{R}^d$ a linear subspace such that* $\dim X \leq d-2$ *holds. Consider an open and path-connected subset $C \subseteq \mathbb{R}^d$ and $x, y \in C \setminus X$. There exists a path $p\colon [0,1] \to C \setminus X$ connecting x and y. In particular $C \setminus X$ is path-connected.*

Proof. Without loss of generality we may assume $X = \mathbb{R}^{d-m} \times \{0\}$ and $m \geq 2$. The set C is path-connected, whence there is a path $q\colon [0,1] \to C$ with $q(0) = x$ and $q(1) = y$. If the intersection $\operatorname{Im} q \cap X$ is empty, there is nothing to prove. Otherwise we construct a path as follows:
Consider the projections $\pi_X\colon \mathbb{R}^d \to \mathbb{R}^{d-m} \times \{0\} = X$ respectively $\pi_2\colon \mathbb{R}^d \to \{0\} \times \mathbb{R}^m$. The projections are continuous open maps, with $\pi_X + \pi_2 = \mathrm{id}_{\mathbb{R}^d}$. Observe that $z \in X$ if and only if $\pi_2(z) = 0$ holds. The set $\{ q(x) | x \in [0,1], \pi_2(q(x)) = 0 \} = \operatorname{Im} q \cap X$ is compact and does not contain x and y. Therefore we may choose $x_i \in X, 1 \leq i \leq N$ and $\varepsilon > 0$ with

$$\operatorname{Im} q \cap X \subseteq \bigcup_{1 \leq i \leq N} B_\varepsilon(x_i) \times B_\varepsilon(0) \subseteq K := \overline{\bigcup_{1 \leq i \leq N} B_\varepsilon(x_i) \times B_\varepsilon(0)} \subseteq C \setminus \{x, y\}.$$

As each closed ball is path-connected the sets $\overline{B_\varepsilon(x_i) \times B_\varepsilon(0)}$ are path-connected. Hence the set K is a set with finitely many path-components $K_1, \ldots, K_r$ (cf. [19, p. 115]). Each path-component is a union $K_i = \bigcup_{1 \leq j \leq r_i} \overline{B_\varepsilon(x_{i_j}) \times B_\varepsilon(0)}$ and is thus compact. Furthermore the boundary ∂K satisfies $\partial K = \partial K_1 \cup \partial K_2 \cup \ldots \cup \partial K_r$, since these sets form a finite partition of closed and disjoint sets. As $\operatorname{Im} q \cap X \subseteq K^\circ$ holds, we deduce that the boundary ∂K_i may not contain elements of $\operatorname{Im} q \cap X$. We construct the path by induction: The set $L_1 := q^{-1}(K_1)$ is a closed subset of $[0,1]$, which does not contain $0, 1$ by construction. By compactness of L_1 we may consider $s_1 := \min\{ x \in L_1 \}$ and $t_1 := \max\{ x \in L_1 \}$. For $t \in \{ s_1, t_1 \}$ we must have $q(t) \in \partial K_1$. Hence by the argument above, $q(s_1), q_(s_2) \notin X$, i.e. $\pi_2(q(s)), \pi_2(q(t_1)) \in \overline{B_\varepsilon(0)} \setminus \{0\}$ holds. Recall that $\overline{B_\varepsilon(0)} \setminus \{0\}$ is path-connected by a variation of [19, V. Theorem 2.2] since $m \geq 2$ is satisfied. Furthermore $\pi_X(K_1)$ is path-connected, whence there is a path $\gamma_1\colon [s_1, t_1] \to \pi_X(K_1) \times \overline{B_\varepsilon(0)} \setminus \{0\} \subseteq K_1 \subseteq C$ with $\gamma_1(s_1) = q(s_1)$ and $\gamma_1(t_1) = q(t_1)$. Define a mapping

$$q_1\colon [0,1] \to C,\ t \mapsto \begin{cases} q(t) & t \in [0,1] \setminus]s_1, t_1[\\ \gamma_1(t) & t \in [s_1, t_1] \end{cases}$$

By construction q_1 is a path with $q(0) = x$ and $q(1) = y$. Furthermore $\operatorname{Im} q_1 \cap K_1 = q_1([s_1, t_1])$ implies $\operatorname{Im} q_1 \cap K_1 \cap X = \emptyset$. In particular the definition of q_1 yields $\operatorname{Im} q_1 \cap X \subseteq \bigcup_{2 \leq r \leq N} K_r$.
Assume that for $1 \leq i < n \leq N$ we have already constructed a path q_i connecting x and y, whose image is contained in C with $\operatorname{Im} q_i \cap X \subseteq \bigcup_{i+1 \leq r \leq N} K_r$. Consider the compact set $L_n := (\alpha^{-1} \circ q_{n-1})^{-1}(K_n) \subseteq]0,1[$. If L_n is empty simply set $q_n := q_{n-1}$ to obtain a path with the desired properties. Otherwise we have to construct a path q_n from q_{n-1}, such that the image does not

intersect $(K_1 \cup \ldots \cup K_n) \cap X$. Apply the above construction verbatim with $L_n \neq \emptyset$ and q_{n-1} instead of L_1 and q. Since q_{n-1} does not intersect $K_i \cap X$ for each $1 \leq i \leq n-1$, the construction yields a mapping q_n with $\operatorname{Im} q_n \cap X \subseteq \bigcup_{n+1 \leq i \leq N} K_i$. holds and its image is contained in C. Summing up, after finitely many steps the mapping $p := q_N$ satisfies: $\operatorname{Im} p \subseteq C$, $p(0) = x, p(1) = y$ and $\operatorname{Im} p \cap X \subseteq \bigcup_{N+1 \leq i \leq N} K_i = \emptyset$. Hence p is a path with the desired properties. □

A.2 Lemma *Let $d, m \in \mathbb{N}$, C be an open connected subset of $\mathbb{R}^d$ and $\{ X_i | i = 1, \ldots, m \}$ a family of vector subspaces of $\mathbb{R}^d$ with $X_i \neq X_j$ for $i \neq j$.*

(a) *For each pair $x, y \in C \setminus \bigcup_{i=1}^m X_i$ there is a path $p\colon [0,1] \to C$ such that*
   1. *$p(0) = x$, $p(1) = y$,*
   2. *$p([0,1]) \cap X_i = \emptyset$ if $\dim X_i \leq n-2$,*
   3. *$p([0,1]) \cap X_i \cap X_j = \emptyset$ if $i \neq j$.*

(b) *Let $k \in \mathbb{N}_0$ with with $\dim X_i = n-1$ if $1 \leq i \leq k$ holds and $\dim X_i < n-1$ otherwise, the set $\mathbb{R}^d \setminus \bigcup_{i=1}^m X_i$ with the subspace topology there has 2^k (path-)connected components.*

(c) *If $C \subseteq \mathbb{R}^d$ is a convex open subset, then $C \setminus \bigcup_{i=1}^m X_i$ possesses at most 2^k connected components.*

Proof. (a) Since for $i \neq j$ we have $\dim X_i \cap X_j \leq n-2$, it suffices to construct a path p which satisfies Properties 1. and 2. for an arbitrary finite number of subspaces Y_i with $\dim Y_i \leq n-2$. Since C is path-connected, $C \setminus Y_1$ is path-connected by Lemma A.1. Iteratively $C \setminus Y_1 \setminus Y_2 \setminus \cdots \setminus Y_m = C \setminus (Y_1 \cup \ldots \cup Y_m)$ is path connected by Lemma A.1.

(b) The subspaces X_i are closed in $\mathbb{R}^d$, whence $\Omega := \mathbb{R}^d \setminus \bigcup_{i=1}^m X_i$ is an open set. The components of Ω coincide with the path-components of Ω by [19, V. 5.6]. We claim that there are 2^k path-components. For a hyperplane X_j we consider the two half spaces H_j^+, H_j^- such that $\mathbb{R}^d$ is the disjoint union $H_j^+ \cup X_j \cup H_h^-$. The half-spaces are the path-components of $\mathbb{R}^d \setminus X_j$. Each half-plane is a convex set. We observe that each intersection of half spaces $H_1^{\sigma(1)} \cap \ldots \cap H_k^{\sigma(k)}$ with $\sigma\colon \{1, 2, \ldots, k\} \to \{+, -\}$ is again a convex set. From (a) we deduce, that these sets yield path-connected subsets of $\mathbb{R}^d \setminus \bigcup_{1 \leq j \leq m} X_j$ if we remove $\bigcup_{k+1 \leq j \leq m} X_j$. Thus the number of components does not change if a subspace X_i with $\dim X_i \leq n-2$ is removed from $\mathbb{R}^n \setminus \bigcup_{l=1}^{i-1} X_l$. On the other hand the number doubles if a hyperplane X_j is removed from $\mathbb{R}^n \setminus \bigcup_{l=1}^{j-1} X_l$. Hence the assertion holds.

(c) From the proof of (b) we deduce that the components are induced by intersections of k half-spaces, which are convex sets. However the same holds for the subset $C \cap H_{j_1}^+ \cap \ldots \cap H_{j_r}^+ \cap H_{j_{r+1}}^- \cap \ldots \cap H_{j_k}^-$. From part (a) we deduce with arguments as in (b), that all non-empty sets of this kind induce the connected components of $C \setminus \bigcup_{i=1}^m X_i$. As there are at most 2^k non-empty sets of this kind, the assertion follows.

□

B. Group actions and Newman‘s theorem

In this section we recall several basic facts concerning group actions, orbit spaces and quotient mappings to orbit spaces. We are interested only in continuous group actions, whence each group action in this paper will be required to be continuous. Several basic results will be repeated for the readers convenience and to fix some notation. For further information on group actions and transformation groups we recommend [11, 54].

B.1. Group actions

B.1.1 Definition (Group actions of topological groups) Let G be a topological group and X a topological space. A *G-action* of G on X is a continuous map $\Theta\colon G \times X \to X$, such that:

(a) $\Theta(\mathbf{1}, x) = x$ for all $x \in X$, where $\mathbf{1}$ is the identity element of G.
(b) $\Theta(g_2, \Theta(g_1, x)) = \Theta(g_2 g_1, x)$ for all $g_1, g_2 \in G$ and $x \in X$.

The pair (X, Θ) is called *G-space* and we denote it usually just by the underlying space X. We shall abbreviate $g.x := \Theta(g, x)$ if it is clear which action is meant. .
For $x \in X$ the *orbit* of x is the set $G.x := \{ g.x | g \in G \}$. Let $X/G := \{ G.x \} x \in X$ be the set of all orbits and endow it with the quotient topology induced by $p\colon X \to X/G, x \mapsto G.x$. The space X/G is called *orbit space* of the G-space X.

B.1.2 Definition (Isotropy Subgroups and Fixed point sets) Let X be a G-space. Define the *isotropy group* $G_x := \{ g \in G | g.x = x \}$ of $x \in X$.
For $g \in G$ the *set of fixed points of* g will be denoted by $\Sigma_g = \{ x \in X | g.x = x \}$, and we write

$$\Sigma_G := \{ x \in X | G_x \neq \{ \mathbf{1} \} \} = \bigcup_{g \in G \setminus \{ \mathbf{1} \}} \Sigma_g$$

For a subset $S \subseteq X$ we define $g.Y := \{ g.x | x \in Y \}$ and let $G_S := \{ g \in G | g.S = S \}$ be the *isotropy group of S*. A subset $Y \subseteq X$ is called *G-invariant* if $G_S = G$ holds. Furthermore a *G-stable* subset S of X is a connected set, such that for $g \in G$ either $g.S = S$ or $g.S \cap S = \emptyset$ is satisfied.

The elegant proof of the following Lemma has been communicated to the author by A. Pohl:

B.1.3 Lemma *Let X be a manifold, G a finite topological group acting on X via homeomorphisms, i.e. $\Theta(g, \cdot)\colon X \to X$ is a homeomorphism for each $g \in G$. For each $x \in X$ there exist arbitrarily small open G-stable neighborhoods of x, whose isotropy subgroups coincide with G_x. In particular the G-stable open sets form a base for the topology on X.*

Proof. Let U be any neighborhood of x and $G.x = \{ x_1, x_2, \ldots, x_n \}$ be the distinct elements in the G-orbit of x, i.e. $x_i \neq x_j$ for $i \neq j$. Without loss of generality $x = x_1$ holds. For $i = 1, \ldots, n$ choose

an open neighborhood U_i of x_i with the following properties: Ffor $i \neq j$ the sets U_i and U_j are disjoint and $U_1 \subseteq U$ holds. For $i = 1, \ldots, n$ define $G_i^1 := \{ g \in G | g.x_i = x \}$ and set

$$S' := \bigcap_{1 \leq i \leq n} \bigcap_{g \in G_i^1} g.U_i.$$

As G acts by homeomorphisms, the set $S' \subseteq U_1 \subseteq U$ is an open neighborhood of x. Consider $h \in G$. If $h.x = x_i$ holds, this implies $h^{-1} \in G_i^1$. Therefore $S' \subseteq h^{-1}.U_i$ yields $h.S' \subseteq U_i$. For $i \neq 1$ we deduce from $U_i \cap U_1 = \emptyset$ and $S' \subseteq U_1$ for h as above $h.S' \cap S' = \emptyset$. On the other hand for $i = 1$ we have $h \in G_x$ and thus

$$h.S' = \bigcap_{j=1}^{n} \bigcap_{g \in G_j^1} (hg).U_j = \bigcap_{j=1}^{n} \bigcap_{g \in G_j^1} g.U_j = S' \tag{B.1.1}$$

Let S be the connected component of S' which contains x. As X is locally path connected S is an open neighborhood of x by [19, V. Theorem 4.2]. Since G acts by homeomorphisms, by (B.1.1) G_x permutes the connected components of S' and fixes x. Combine (B.1.1) and the fact $h.S' \cap S' = \emptyset$ for $h \in G \setminus G_x$. We deduce that $G_S = G_x$ holds and S is a G-stable open neighborhood of x which is contained in $S' \subseteq U$. □

B.1.4 Lemma ([54, Proposition 3.1 and Proposition 3.6]) *Let X be a Hausdorff G-space and G a compact topological group. Consider the quotient map $\pi \colon X \to X/G, x \mapsto G.x$ onto the orbit space.*

(a) *X/G is a Hausdorff space.*
(b) *π is a continuous, open and closed map.*
(c) *π is a proper map.*
(d) *X is compact if and only if X/G is compact.*
(e) *X is locally compact if and only if X/G is locally compact.*

B.1.5 Remark Let $\mathrm{Diff}^r(M)$ be the group of C^r-diffeomorphisms from a C^r-manifold M to itself for $r \in \mathbb{N}_0 \cup \{ \infty \}$. To shorten the notation we write $\mathrm{Diff}(M) := \mathrm{Diff}^\infty(M)$. The discrete topology is the unique Hausdorff topology turning a finite subgroup G of $\mathrm{Diff}^r(M)$ into a topological group. Any finite group $G \subseteq \mathrm{Diff}^r(M)$ becomes a compact topological group in this way. The natural mapping $\Theta \colon G \times M \to M, (g, x) \mapsto g(x)$ is continuous since each element in G is continuous and G is endowed with the discrete topology. Hence each finite subgroup G of $\mathrm{Diff}^r(M)$ induces a canonical action of a compact topological group on M. Furthermore the quotient map $M \to M/G$ with respect to such a G-action satisfies the prerequesits of Lemma B.1.4.

B.1.6 Definition Let $f \colon X \to Y$ be a map, G a group, X a G-space and Y be a topologcal space.

(a) If Y is a G-spaces, we call f *equivariant* if $f(g.x) = g.f(x)$ holds for all $x \in X$ and $g \in G$.
(b) Let H be another topological group such that Y is a H-space. If there is a group homomorphism $\lambda \colon G \to H$ such that $f(g.x) = \lambda(g).f(x)$ holds for all $x \in X, g \in G$, f is called *equivariant with respect to λ*.

B.2. Newman‘s Theorem

The Theorem due to M.H.A. Newman discussed in this section is an important tool to investigate the structure of orbifolds (for a proof see [18]):

B.2.1 Theorem (Newman 1931) *Let G be a finite group acting effectively by homeomorphisms on a connected manifold M, then the set $M \setminus \Sigma_G$ of points with trivial isotropy group is a dense and open set.*

In the situation of Theorem B.2.1, the elements of Σ_G are called *singular points* and the elements of $M \setminus \Sigma_G$ are called *non-singular points*. We compile several interesting consequences of Newman's Theorem. For further information we refer to [48, Section 2.4].

B.2.2 Lemma (cf. [48, p. 36]) *Let M be a smooth finite dimensional paracompact manifold, G a finite subgroup of* $\mathrm{Diff}(M)$ *and $x \in M$. Then there exists an arbitrarily small G-stable chart (W, κ) with $x \in W$ such that κ conjugates the isotropy group G_x to a (finite) group of orthogonal transformations on $\kappa(W)$. Furthermore $T_x g = \mathrm{id}_{T_xM}$ implies $g|_W = \mathrm{id}_W$ for each $g \in G_x$.*

Proof. Since G is finite we may choose a G-invariant Riemannian metric on M by [48, Proposition 2.8]. The group G thus acts via Riemannian isometries with respect to this metric. Let $\exp_M$ be the Riemannian exponential map with respect to this metric. By [17, 3 Proposition 2.9], $\exp_M$ induces a diffeomorphism from an open ball $B_\varepsilon(0_x)$ centered at 0_x in $T_x(M)$ to an open neighbourhood W of x, $\exp_{M,x} \colon B_\varepsilon(0_x) \to W \subseteq M$. SAs the metric is G-invariant, each $g \in G_x$ induces an orthogonal transformation $T_x g$ of T_xM. Since $\exp_M$ commutes with Riemannian isometries on its domain, we deduce $\exp_{M,x} \circ T_x g|_{\mathrm{dom}\exp_{M,x}} = g \circ \exp_{M,x}$. This formula shows that $T_x g = \mathrm{id}$ implies $g|_W = \mathrm{id}_W$ and W is G_x invariant. By continuity of $\exp_M$ we may shrink ε, such that W is contained in G-stable neighborhood of x (cf. Lemma B.1.3). Hence there is $\varepsilon > 0$, such that $\exp_{M,x}(B_\varepsilon(0)) = W$ is a G-stable subset with $G_W = G_x$. For such a W define $\kappa := (\exp_{M,x}|_W)^{-1}$. The pair (W, κ) satisfies the assertion. In particular W may be taken arbitrarily small. $\square$

B.2.3 Lemma *Let M be a connected paracompact smooth manifold and G be a finite subgroup of* $\mathrm{Diff}(M)$. *Denote by Σ_{TG} the set of of singular points with respect to the derived action $G \times TM \to TM, (g, X) \mapsto g.X := Tg(X)$ of G on TM. For each open connected set $U \subseteq TM$ the set of non-singular points $U \setminus \Sigma_{TG}$ is (path-)connected.*

Proof. The manifold TM is locally path-connected, whence for open sets connected components and path-components coincide by [19, V. Theorem 5.5]. Let $\mathcal{C}$ be the family of (path-)connected components in $U \setminus \Sigma_{TG}$. To prove the Lemma we have to assert $|\mathcal{C}| = 1$. Suppose that $|\mathcal{C}| > 1$ holds and consider $C \in \mathcal{C}$. The boundary ∂C of C with respect to U is contained in $\Sigma_{TG} \cap U$. Observe that $U \subseteq TM$ is open and path-connected and Σ_{TG} is nowhere dense in TM by Newman's Theorem

B.2.1. Hence ∂C satisfies

$$\partial C \cap \bigcup_{C' \in \mathcal{C} \setminus \{ C \}} \partial C' \neq \emptyset.$$

In other words there are no isolated components of $U \setminus \Sigma_{TG}$, i.e. the boundary of $C \in \mathcal{C}$ intersects the boundaries of other components if there are several connected components of $U \setminus \Sigma_{TG}$.

We claim that $\partial C \cap \partial C' \neq \emptyset$ (in U) forces both components to coincide. Since there are no isolated components, if the claim were true $U \setminus \Sigma_{TG}$ is connected. To prove our claim, consider $C, C' \in \mathcal{C}$ with $X := (\pi_{TM}(X), \xi) \in \partial C \cap \partial C'$. By definition of the derived action for $g \in G$ we have $Tg(X) = g.X = X$ if $g.\pi_{TM}(X) = \pi_{TM}(X)$. This implies $G_X \subseteq G_{\pi_{TM}(X)}$. By Lemma B.2.2 there is a G-stable manifold-chart (W, κ) such that $\pi_{TM}(X) \in W \subseteq U$ and κ conjugates $G_{\pi_{TM}(X)}$ to a finite group of orthogonal transformations on $\kappa(W) = B_\varepsilon(0_{\pi_{TM}(X)}) \subseteq \mathbb{R}^d$ for $d = \dim M$ and some $\varepsilon > 0$. Identify TW with an open subset of TU to obtain a chart $(TW, T\kappa)$ for TU with $X \in TW$. Since $X \in \partial C \cap \partial C'$ the open set TW intersects both components. Hence $TW \setminus \Sigma_{TG}$ intersects both components. If $TW \setminus \Sigma_{TG}$ is a connected set, then $C = C'$ follows. To prove this, identify $T\kappa(TW) = TB_\varepsilon(0_{\pi_{TW}(X)})$ with $B_\varepsilon(0) \times \mathbb{R}^d \subseteq \mathbb{R}^{2d}$. For $g \in G_{\pi_{TW}(X)}$ let $\tilde{g}$ be the orthogonal transformation conjugate to g, i.e. $\tilde{g}$ is a linear map with $\tilde{g}\kappa = \kappa g$. The functoriality of T implies $T\tilde{g}T\kappa = T\kappa Tg$. Taking identifications $T\tilde{g} = (\tilde{g}|_{B_\varepsilon(0)}, d\tilde{g}) = (\tilde{g}|_{B_\varepsilon(0)}, \tilde{g} \circ \mathrm{pr}_2)$ is the restriction of a linear map. Thus $T\kappa$ conjugates the action of $G_X \subseteq G_{\pi_{TM}(X)}$ on TW to a linear action on $T\kappa(TW) = B_\varepsilon(0) \times \mathbb{R}^d$. Since W is G-stable with $G_W = G_{\pi_{TM}(X)}$, the set TW is G-stable with $G_{TW} = G_{\pi_{TM}(X)}$ by definition of the derived action. Hence $TW \cap \Sigma_{TG} \subseteq TW \cap \Sigma_{TG_{\pi_{TW}(X)}}$ holds. Therefore $\tilde{V} := T\kappa(TW \cap \Sigma_{TG}) = (B_\varepsilon(0) \times \mathbb{R}^d) \setminus T\kappa(\Sigma_{TG_{\pi_{TM}(X)}})$ holds. We claim that $\tilde{V}$ is connected. If this were true, the same holds for $TW \setminus \Sigma_{TG}$, whence the proof is complete.
Proof of the claim: As $T\kappa$ conjugates the group action to a linear action, the set $T\kappa(TW \cap \Sigma_{TG})$ is a finite union of linear subspaces of $\mathbb{R}^{2d}$. By Lemma A.2 the set $\tilde{V}$ will be connected, if and only if for each $g \in G_X$ the fixed point set of the associated linear map $T\tilde{g}$ is not a hyperplane in $\mathbb{R}^{2d}$. For each $g \in G_{\pi_{TM}(X)} \setminus \{ \mathrm{id}_M \}$ a combination of [48, Lemma 2.10 and Lemma 2.11] implies that $\tilde{g}$ is not the identity map. From [12, Proposition 2.18 (1)] we deduce that the fixed points of $\tilde{g}$ are contained in a hyperplane $H \subsetneq \mathbb{R}^d$. Each linear subspace fixed by $T\tilde{g}$ is thus contained in $H \times H$ and $\dim H \times H = d - 2$. Hence $T\tilde{g}$ does not fix any hyperplane, whence $T\kappa(TW \setminus \Sigma_{TG})$ is connected. □

C. Infinite dimensional manifolds and Lie groups

In this section we briefly recall the notion of infinite dimensional manifolds and infinite dimensional Lie groups. Manifolds and Lie groups modelled on infinite dimensional spaces may be defined almost exactly as in the finite dimensional case.

C.1. Manifolds modelled on locally convex spaces

C.1.1 Definition We recall from [30] that a *manifold with rough boundary* modelled on a locally convex space E is a Hausdorff topological space M with an atlas of smoothly compatible homeomorphisms $\phi\colon V_\phi \to U_\phi$ from open subsets V_ϕ of M onto locally convex subsets $U_\phi \subseteq E$ with dense interior. If each U_ϕ is open, M is an ordinary manifold (without boundary). In a similar fashion C^r-manifolds may be defined for $r \in \mathbb{N}_0$. Unless stated otherwise, every manifold will be assumed to be without boundary. Direct products of locally convex C^k-manifolds, tangent spaces and tangent manifolds may be defined as in the finite dimensional setting. We refer to [50] for details.

C.1.2 Notation Let M, N be C^r-manifolds (where $1 \leq r \leq \infty$), and $f\colon M \to N$ a mapping of class C^r. We denote by $Tf\colon TM \to TN$ the tangent map. Abbreviate by $T_xf\colon T_xM \to T_{f(x)}N$ the restriction of Tf to the tangent space T_xM of M at $x \in M$. If N is a locally convex space the tangent map $Tf\colon TM \to TN \cong N \times N$ is given by $(x, v) \mapsto (f(x), df(x, v))$ for $x \in M$, $v \in T_xM$ and a map $df\colon TM \to N$. If $f\colon U \to V$ is a C^r-map, where U, V are open subsets of locally convex spaces E and F, it is convenient to think of $df(x, y)$ as a tangential map. Hence we canonically identify $T_xU \cong E$ and $T_yV \cong F$ to obtain $df(x, v) = T_xf(v)$.
We let $\pi_{TM}\colon TM \to M$ be the bundle projection. For $r = \infty$ we denote by $\mathfrak{X}(M)$ the space of smooth vector fields, i.e. smooth mappings $X\colon M \to TM$ with $\pi_{TM} \circ X = \mathrm{id}_M$.

C.2. Function spaces and their topologies

Our exposition of the C^r-topology follows [25], but we allow locally convex subsets. Albeit the definition of differentiability differs from the one used in [25], on open subsets of locally convex spaces over the field $\mathbb{R}$ both coincide by [5, Proposition 7.4].

C.2.1 Definition (Compact open topology) Let X, Y be Hausdorff topological spaces, $K \subseteq X$ compact and $U \subseteq Y$ open. We define the set

$$\lfloor K, U\rfloor := \{\, f \in C(X, Y) | f(K) \subseteq U \,\}$$

Then the sets

$$\lfloor K_1, U_1\rfloor \cap \lfloor K_2, U_2\rfloor \cap \ldots \cap \lfloor K_n, U_n\rfloor$$

with $n \in \mathbb{N}$, $K_i \subseteq X$ compact and $U_i \subseteq Y$ open for $1 \leq i \leq n$, are a base for a topology on $C(X, Y)$ (cf. [20, Section 3.4]). It is called the *compact-open topology* and we denote by $C(X, Y)_{c.o.}$ the space $C(X, Y)$ with this topology .

C.2.2 Definition Let E, F be locally convex topological vector spaces and $U \subseteq E$ a locally convex subset with dense interior, $r \in \mathbb{N}_0 \cup \{ \infty \}$. Endow $C^r(U, F)$ with the unique locally convex topology turning

$$(d^{(j)}(\cdot))_{\mathbb{N}_0 \ni j \leq r} \colon C^r(U, F) \to \prod_{0 \leq j \leq r} C(U \times E^j, F), f \mapsto (d^{(j)} f)$$

into a topological embedding. We call this topology the *compact-open C^r-topology*. Notice that it is the initial topology with respect to the family $(d^{(j)}(\cdot))_{\mathbb{N} \ni j \leq r}$.

C.2.3 Remark

(a) By [22, Lemma 1.14], Definition C.2.2 coincides on open sets with the definition in [23, Definition 3.1]. Hence if U is an open, σ-compact and locally compact subset of a locally convex space E and F is a Fréchet space, then $C^r(U, F)$ is a Fréchet space by [23, Remark 3.2].

(b) For each compact subset $K \subseteq U$ and open subset $V \subseteq F$, the set

$$\lfloor K, V \rfloor_r := \{ \gamma \in C^r(U, F) | \gamma(K) \subseteq V \}$$

is open in $C^r(U, F)$ by [25, Lemma 4.22].
If $s, r \in \mathbb{N}_0 \cup \{ \infty \}$ with $r \leq s$, by definition then $C^s(U, F) \subseteq C^r(U, F)$ holds and the topology on $C^s(U, F)$ is finer than the subspace topology induced by $C^r(U, F)$. Let Ω be an open set in $C^s(U, F)$, such that $\Omega = C^s(U, F) \cap A$ holds for some open $A \subseteq C^r(U, F)$. Then we call Ω *a C^r-open set in $C^s(U, F)$* or a *C^r-neighborhood of $f \in C^s(U, F)$*.

C.2.4 Definition Let E be a locally convex vector space and M a C^r-manifold. Then we let $C^r(M, E)$ be the space of all C^r-mappings $\gamma \colon M \to E$. The pointwise operations turn $C^r(M, E)$ into a vector space. Endow $C^r(M, E)$ with the initial topology with respect to the family

$$\theta_\kappa \colon C^r(M, E) \to C^r(V_\kappa, E), \gamma \mapsto \gamma|_{U_\kappa} \circ \kappa^{-1}$$

where $\kappa \colon U_\kappa \to V_\kappa$ ranges through an atlas of M. If M is an open subset of a locally convex space, [25, Lemma 4.6] proves that this topology coincides with the compact open C^r-topology.

C.2.5 Definition

(a) Let $U \subseteq \mathbb{R}^n$ be some open subset $n \in \mathbb{N}_0$ and $K \subseteq U$ compact. For $\xi \in C^r(U, \mathbb{R}^n)$, $r \in \mathbb{N}_0 \cup \{ \infty \}$ and $k \in \mathbb{N}_0$ with $k \leq r$, we use standard multiindex notation and set

$$\|\xi\|_{K,k} := \max_{|\alpha| \leq k} \max_{x \in K} \|\partial^\alpha \xi(x)\|$$

(b) Let E be a locally convex space and $r \in \mathbb{N}_0 \cup \{ \infty \}$. Endow $C^r([0,1], E)$ with the locally convex vector topology induced by the family of seminorms $\|\cdot\|_{C^k,p}$ defined via

$$\|\gamma\|_{C^k,p} := \max_{j=0,\ldots,k} \max_{t \in [0,1]} p\left(\frac{\partial^k}{\partial t^k} \gamma(t)\right)$$

where p ranges through the continuous seminorms on E and $k \in \mathbb{N}_0$ with $k \leq r$.

C.2.6 Remark

i. Let $U \subseteq \mathbb{R}^n$ be some open subset $n \in \mathbb{N}_0$. As U is σ-compact, there is a sequence of compact sets $(K_n)_{\mathbb{N}}$ such that $U = \bigcup_{n\in\mathbb{N}} K_n$. By a variant of [23, Proposition 4.4], the locally convex topology induced by the family of seminorms $\left\{ \|\cdot\|_{K_n,k} \middle| n \in \mathbb{N}, 0 \leq k \leq r \right\}$ on $C^r(U,\mathbb{R}^n)$ coincides with the compact open C^r-topology.

ii. A variant of [23, Proposition 4.4] shows that this topology is initial with respect to the mappings $d^{(j)} \colon C^r(J,E) \to C(J,E)_{\text{c.o}}, \gamma \mapsto d^{(j)}\gamma$, $0 \leq k \leq r$, i.e. it coincides with the compact-open C^r-topology.
In particular $C^r([0,1],U) := \{ \gamma \in C^r([0,1],E) | \gamma([0,1]) \subseteq U \} = \lfloor [0,1], U \rfloor_r$ is an open subset for each $U \subseteq E$. If E is metrizable (respectively complete), $C^r([0,1],E)$ is metrizable by [37, 2.8 Theorem 1] (respectively complete by [31, Lemma 1.4]).

C.3. Spaces of sections and patched spaces

In this section we endow the *space of smooth vector fields* $\mathfrak{X}(M)$ on a smooth manifold M with a topology. Furthermore we introduce the concept of a "patched locally convex space" (cf. [24,25]), to obtain a criterion for the differentiability of maps between spaces of sections. We recall the following facts from [25, Appendix F]:

C.3.1 Definition Let M be a smooth manifold modelled on the locally convex space E and $\pi_{TM} \colon TM \to M$ be the bundle projection. Consider a maximal atlas $\mathcal{A}$ of M and a chart $(V_\psi, \psi) \in \mathcal{A}$ with $\psi \colon V_\psi \to U_\psi$. Let $\mathrm{pr}_2 \colon V_\psi \times E \to E$ be the canonical projection.
For a vector field $X \in \mathfrak{X}(M)$, we define a local representative $X_\psi := \mathrm{proj}_2 \circ T\psi \circ X|_{V_\psi} \colon V_\psi \to E$. In particular $X(y) = (y, X_\psi(y))$ holds for all $y \in U_\psi$.
We endow $\mathfrak{X}(M)$ with the unique locally convex topology, turning the linear map

$$\Gamma \colon \mathfrak{X}(M) \to \prod_{(V_\psi,\psi)\in\mathcal{A}} C^\infty(V_\psi, E),\ X \mapsto (X_\psi)_{(V_\psi,\psi)\in\mathcal{A}}$$

a topological embedding. In particular the topology on $\mathfrak{X}(M)$ is the initial topology with respect to the family of linear maps $\theta_\psi \colon \mathfrak{X}(M) \to C^\infty(V_\psi, E), X \mapsto X_\psi$.

C.3.2 Lemma ([25, Lemma F.9]) *The topology on $\mathfrak{X}(M)$ is initial with respect to the family $(\theta_\phi)_{(V_\phi,\phi)\in\mathcal{B}}$, where $\mathcal{B} \subseteq \mathcal{A}$ is some atlas for M.*

Proof. Combine [25, Lemma F.9] with [25, Proposition 4.19], which guarantees that the topology defined in [25] coincides with our definition of the compact open C^r-topology over the field $\mathbb{R}$. ☐

C.3.3 Notation Let M be a smooth manifold and U an open subset of M. We define the restriction map $\mathrm{res}^M_U \colon \mathfrak{X}(M) \to \mathfrak{X}(U), X \mapsto X|_U$. For each open subsete U this maps is continuous linear by [25, Lemma F.15].

C.3.4 Definition A *patched locally convex space* over $\mathbb{R}$ is a pair $(E, (p_i)_{i\in I})$, where E is a topological $\mathbb{R}$-vector space and $(p_i)_{i\in I}$ is a family of continuous linear maps $p_i\colon E \to E_i$ to topological vector spaces E_i, such that

(a) for each $x \in E$, the set $\{\, i \in I | p_i(x) \neq 0 \,\}$ is finite,

(b) the linear map
$$p\colon E \to \bigoplus_{i\in I} E_i,\ x \mapsto (p_i(x))_{i\in I} = \sum_{i\in I} p_i(x)$$
from E to the direct sum $\oplus_{i\in I} E_i$ (equipped with the direct sum topology cf. [10, II.29.5 Definition 2]) is a topological embedding,

(c) the image $p(E)$ is sequentially closed in $\bigoplus_{i\in I} E_i$.

The mappings $p_i\colon E \to E_i$ are called *patches* , and the family $(p_i)_{i\in I}$ is called a *patchwork* . If I is a countable set, we also say that E is *countably patched*.

C.3.5 Lemma *Let $(E, (p_i)_{i\in I})$ be patched topological $\mathbb{R}$-vector space, with $p_i\colon E \to E_i$ and p as in Definition C.3.4. For each $r \in \mathbb{N}_0 \cup \{\,\infty\,\}$, the map*
$$p_*\colon C^r([0,1], E) \to C^r([0,1], \bigoplus_{i\in I} E_i), g \mapsto p \circ g$$
is a linear topological embedding whose image is sequentially closed. If E is countably patched and $|I| < \infty$ or $r < \infty$ holds, the family $C^r([0,1], p_i)\colon C^r([0,1], E) \to C^r([0,1], E_i), \gamma \mapsto p_i \circ \gamma,\ i \in I$, turns $C^r([0,1], E)$ into a patched topological $\mathbb{R}$-vector space.

Proof. The maps $(C^r([0,1], p_i))_{i\in I}$ are continuous linear and p_* is a topological embedding by [31, Lemma 1.2]. Without loss of generality we identify E with a subspace of $F := \bigoplus_{i\in I} E_i$. Let $(f_n)_{n\in\mathbb{N}} \subseteq \operatorname{Im} p_*$ be a sequence which converges to some $f \in C^r([0,1], F)$. Since E is sequentially closed, due to the continuity of the point evaluation maps (cf. [2, Proposition 3.20]) for $t \in [0,1]$ the sequence $(f_n(t))_{n\in\mathbb{N}}$ converges in E. Hence the image of f is contained in E. Recall that directional derivatives may be computed as limits of sequences. As each element $f(t) = \lim_{n\in\mathbb{N}} f_n(t)$ is contained in E and E is sequentially close, the mappings $d^{(k)}f$ $0 \leq k \leq r$ take their images in E. Hence $f \in C^r([0,1], E)$ holds and $\operatorname{Im} p*$ is sequentially closed as a subspace of $C^r([0,1], F)$.

First case: $|I| < \infty$. Since I is finite, the coproduct F in the category of locally convex topological vector spaces coincides with the product of the E_i. Hence the canonical projection $\pi_i\colon F \to E_i$ and the canonical inclusion $\iota_i\colon E_i \to F$ are continuous linear for $i \in I$. From [31, Lemma 1.2] we deduce that the mappings
$$((\pi_i)_{i\in I})_*\colon C^r([0,1], \bigoplus_{i\in I} E_i) \to \bigoplus_{i\in I} C^r([0,1], E_i), f \mapsto (\pi_i \circ f)_{i\in I},$$
$$\bigoplus_{i\in I} C^r([0,1], E_i) \to C^r([0,1], \bigoplus_{i\in I} E_i), (f_i) \mapsto \sum_{i\in I} (\iota_i)_*(f_i)$$
are continuous linear and mutually inverse. Thus $C^r([0,1], \bigoplus_{i\in I} E_i)$ and $\bigoplus_{i\in I} C^r([0,1], E_i)$ are isomorphic as locally convex spaces, whence the maps $(p_i)_*, i \in I$ form a patchwork for $C^r([0,1], E)$.

Second case: $|I| = \infty$ **and** $r < \infty$. The canonical inclusions yield a family of continuous linear maps $((\iota_i)_*)_{i\in I}$ by [31, Lemma 1.2]. As in the first case we obtain a linear and continuous map $\Lambda\colon \bigoplus_{i\in I} C^r([0,1],E_i) \to F, (\sum_{i\in I}\gamma_i) \mapsto \sum_{i\in I}(\iota_i)_*(\gamma_i)$. For the rest of the proof we suppress the inclusions ι_i in the notation. To prove our claim we have to construct an inverse mapping for Λ. To do so pick $\gamma \in C^r([0,1],F)$. The compact set $\gamma([0,1]) \subseteq F$ is contained in a finite partial sum by [10, 3, III.4 §1, Proposition 5]. As the inclusion of a finite partial sum is a topological embedding, from [31, Lemma 1.2] and the isomorphism established for the finite case, we deduce that there are unique $\gamma_i \in C^r([0,1],E_i)$, $\forall i \in I$ with $\gamma = \Lambda((\gamma_i)_{i\in I})$. Hence we obtain a well-defined inverse of Λ via $\Theta\colon C^r([0,1],F) \to \bigoplus_{i\in I} C^r([0,1],E_i), \gamma \mapsto (\gamma_i)_{i\in I}$.
We claim that Λ is an isomorphism of locally convex spaces. To prove the clim let Γ_i be the set of all continuous seminorms on E_i. Consider $q = (q_i)_{i\in I} \in \Gamma := \prod_{i\in I}\Gamma_i$ and obtain a continuous seminorm $r_q\colon F \to [0,\infty[, r_q(\sum_{i\in I} x_i) := \sup\{\, q_i(x_i) | i \in I\,\}$ with $x_i \in E_i$. Since the space E is countably patched, the topology on F coincides with the box topology by [37, Proposition 4.1.4]. Hence the family $(r_q)_{q\in\Gamma}$ determines the locally convex topology on F. By definition of the topology on $C^r([0,1],F)$, the continuous seminorms $s_Q\colon C^r([0,1],F) \to [0,\infty[$,

$$s_q(\gamma) := \sup_{0\le k\le r}\ \sup_{x\in[0,1]} r_q(\tfrac{\partial^k}{\partial x^k}\gamma(x)) = \sup_{0\le k\le r}\ \sup_{x\in[0,1]}\ \sup_{i\in I} q_i(\tfrac{\partial^k}{\partial x^k}\gamma_i(x)),$$

where q ranges through Γ determine the locally convex topology on $C^r([0,1],F)$. In the same fashion we deduce that the locally convex topology on $C^r([0,1],E_i)$ is determined by the continuous seminorms $t_{q_i}\colon C^r([0,1],E_i) \to [0,\infty[, t_{q_i}(\gamma_i) := \sup_{0\le k\le r}\sup_{x\in[0,1]} q_i(\frac{\partial^k}{\partial x^k}\gamma_i(x))$, where q_i through Γ_i. The locally convex sum topology, i.e. the box topology on $\bigoplus_{i\in I} C^r([0,1],E_i)$ is induced by the family of seminorms $u_q\colon \bigoplus_{i\in I} C^r([0,1],E_i) \to [0,\infty[$,

$$u_q((\gamma_i)_{i\in I} := \sup_{i\in I} t_{q_i}(\gamma_i) = \sup_{i\in I}\ \sup_{0\le k\le r}\ \sup_{x\in[0,1]} q_i(\tfrac{\partial^k}{\partial x^k}\gamma_i(x))$$

for $q = (q_i)_{i\in I} \in \Gamma$. Observe that for each $q \in \Gamma$ we have $s_q \circ \Lambda = u_q$. We deduce that Λ^{-1} is continuos (cf. [10, II, §2 No. 4 Proposition 4]), whence Λ is an isomorphism of locally convex spaces. □

If $r = \infty$ and $|I| = \infty$, the map Λ introduced in the proof of Lemma C.3.5 still is a continuous linear bijection, but its inverse fails to be continuous in general.

C.3.6 Definition Let I be some set and $(E,(p_i)_{i\in I})$ and $(F,(q_i)_{i\in I})$ patched topological $\mathbb{R}$-vector spaces with canonical embeddings $p\colon E \to \bigoplus_{i\in I} E_i$ and $q\colon F \to \bigoplus_{i\in I} F_i$ as in Defintion C.3.4.

(a) A map $f\colon U \to F$ defined on an open subset $U \subseteq E$ is called *patched mapping* if there exists a family $(f_i)_{i\in I}$ of mappings $f_i\colon U_i \to F_i$ on certain open neighborhoods U_i of $p_i(U)$ in E_i, which is *compatible with* f in the following sense: We have $0 \in U_i$ and $f_i(0) = 0$ for all but finitely many i, and $q_i(f(x)) = f_i(p_i(x))$ for all $i \in I$, i.e. $q \circ f = (f_i)_{i\in I} \circ p|_U^{\oplus U_i}$
(b) For $k \in \mathbb{N}_0 \cup \{\,\infty\,\}$, we say that a patched mapping $f\colon U \to F$ is *of class* C^k *on the patches* if all of the mappings f_i in (a) can be chosen of class C^k.

C.3.7 Proposition *Let I be some set and $(E,(p_i)_{i\in I})$, $(F,(q_i)_{i\in I})$ be patched topological $\mathbb{R}$-vector spaces. Assume that $f\colon U \to F$ is a patched mapping from an open subset $U \subseteq E$ to F. If f is of class C^{k+1} on the patches, then f is of class C^k. If E and F are countably patched and f is C^k on the patches, then f is of class C^k.*

Proof. For $i \in I$ let $f_i\colon U_i \to F_i$ be the map compatible with f. Consider the box neighborhood $\oplus_{i\in I} U_i := \left(\prod_{i\in I} U_i\right) \cap \left(\bigoplus_{i\in I} E_i\right)$ which is open in the locally convex sum (cf. [37, 4.3]). The compatibility condition yields $q \circ f = (f_i)_{i\in I} \circ p|_U^{\oplus U_i}$. As shown in [24, Proposition 7.1] the map $(f_i)_{i\in I}$ is a C^k-map if each f_i is of class C^{k+1} (respectively if each f_i is a C^k-map and I is countable). By definition this is the case if and only if f is a C^{k+1}-map (respectively a C^k-map in the countable case) on the patches. The map $(f_i)_{i\in I} \circ p|_U^{\oplus U_i}$ is of class C^k as a composition of a C^k-map and a smooth map. Thus $q\circ f$ is a C^k-map. Since the subspace $\operatorname{Im} q$ is sequentially closed, the corestriction $(q\circ f)|^{\operatorname{Im} q}$ is a C^k map. As $q|^{\operatorname{Im} q}$ is an isomorphism of topological vector spaces, f is a C^k-map. □

C.4. Lie groups

C.4.1 Definition A *(locally convex) Lie group* is a group G equipped with a smooth manifold structure turning the group operations into smooth maps. Denote its neutral element by $\mathbf{1}$ and recall that $L(G) := T_{\mathbf{1}}G$ is its Lie algebra. (cf. [22, 50] for details)

C.4.2 Definition Let G be a Lie group. We denote by $\rho_g\colon G \to G, h \mapsto hg$ the right translation by $g \in G$. This yields a natural action of G on the tangent Lie group TG (cf. [9, III. §2])

$$v \cdot g := (T_x\rho_g)(v) \in T_{gx}G \quad \text{for } x \in G, v \in T_xG$$

The following construction principle for Lie groups will be our main tool to construct Lie group structures. A proof for the Banach case is given in [9, III. §1.9, Proposition 18]. It carries over without any changes to the more general situation:

C.4.3 Proposition *Let G be a group and U, V subsets of G such that $\mathbf{1} \in V = V^{-1}$ and $V \cdot V \subseteq U$. Suppose that U is equipped with a smooth manifold structure, modelled on a locally convex space, with respect to which the maps $\iota\colon V \to V \subseteq U$ and $\mu\colon V \times V \to U$ -induced by inversion and the group multiplication respectively - are smooth. Here we consider V as an open submanifold of U. Then the following holds*

a) There is a unique smooth manifold structure on the subgroup $G_0 := \langle V \rangle$ of G generated by V such that G_0 becomes a Lie group, V is open in G_0, and such that U and G_0 induce the same smooth manifold structure on the open subset V.

b) Suppose that for each g in a generating set of G, there is an open identity neighborhood $W \subseteq U$, such that $c_g\colon W \to U, h \mapsto ghg^{-1}$ is smooth, then there is a unique smooth manifold structure on G turning G into a Lie group, such that V is open in G and both G and U induce the same smooth manifold structure on the open subset V.

C.5. Regular Lie groups

C.5.1 Definition Let G be a Lie group with Lie algebra $L(G)$. Consider a C^k-curve $p\colon [0,1] \to G$ with $k \geq 1$ and recall that

$$\delta^r p \in C^{k-1}([0,1], L(G)), (\delta^r p)(t) := p'(t) \cdot p(t)^{-1}$$

is called the *right logarithmic derivative* of p. Furthermore we call p a *right product integral* for $\delta^r p$. If $q\colon [0,1] \to G$ is another C^k-curve such that $\delta^r p = \delta^r q$ (i.e. both p and q are right product integrals for $\delta^r q$), then $q = g_0 \cdot p$ holds for some constant $g_0 \in G$ (cf. [46, Lemma 7.4])

C.5.2 Definition If $\gamma \in C^k([0,1], L(G))$ with $k \in \mathbb{N}_0 \cup \{\infty\}$ admits a right product integral p, we define $\mathcal{P}(\gamma) := p \cdot p(0)^{-1}$. Thus $\mathcal{P}(\gamma)$ is a right product integral for γ, such that $\mathcal{P}(\gamma)(0) = \mathbf{1}_G$ is the identity element of G. The product integral is uniquely determined by this property.

C.5.3 Definition Let $k \in \mathbb{N}_0 \cup \{\infty\}$. A Lie group G with Lie algebra $L(G)$ is called *(strong) C^k-regular*, if for the unit intervall $[0,1]$ and each $\xi \in C^k([0,1], L(G))$, the initial value problem

$$\gamma(0) = \mathbf{1}_G, \qquad \delta^r(\gamma) = \xi \tag{C.5.1}$$

has a solution $\mathcal{P}(\xi)$, which is then contained in $C^{k+1}([0,1], G)$, and the corresponding evolution map

$$\mathrm{evol}_G\colon C^k([0,1], L(G)) \to G, \xi \mapsto \mathcal{P}(\xi)(1)$$

is smooth. If G is C^k-regular, we write

$$\mathrm{Evol}_G\colon C^k([0,1], L(G)) \to C^{k+1}([0,1], G), \xi \mapsto \mathcal{P}(\xi)$$

for the map on the level of Lie group-valued curves. For more information on regularity see [29]. The group G is called *regular (in the sense of Milnor)* if it is C^∞-regular. For $k \leq r$ the C^r-regularity follows from C^k-regularity.

Notice that we have defined regularity properties of Lie groups using the right logarithmic derivative. Alternatively one may define *left logarithmic derivative*, *left product integrals* and regularity properties using these notions. However, it is well known that this results in the same concepts of regularity as defined in C.5.3. See [16, Proposition 1.3.6] for a proof.

The following Lemma will be our main tool to prove the regularity of the Orbifold diffeomorphism group. Its proof carries over almost verbatim from [16, Proposition 1.3.10]:

C.5.4 Lemma *Let G be a smooth Lie group with Liealgebra $L(G)$. Assume that there is a zero-neighborhood $U \subseteq L(G)$ such that for $k \in \mathbb{N}_0 \cup \{\infty\}$, every $\xi \in C^k([0,1], U)$ has a right product integral. Furthermore assume that $E_1\colon C^k([0,1], U) \to G, \xi \mapsto \mathcal{P}(\xi)(1)$ is smooth. Then G is C^k-regular.*

D. Riemannian geometry and vector fields

In this paper we assume basic familiarity with Riemannian metrics and geodesics. However our approach requires standard results from Riemannian geometry as outlined in [17, 39, 41]. The goal of this section is to fix teh necessary notation and to provide estimates needed in the proof of the main theorems.

D.0.1 Notation The pair (M, ρ_M) will always denote a finite dimensional smooth Riemannian manifold M, with Riemannian metric ρ_M. Notice that for each $x \in M$ the Riemannian metric yields a positive definite inner product $\rho_{M,x}\colon T_xM \times T_xM \to \mathbb{R}$. We usually abbreviate

$$\rho_M(X,Y) := \rho_{M,x}(X,Y) \quad \forall X, Y \in T_xM$$

We define the *ε-balls with respect to the Riemannian metric in T_xM around the origin* 0_x as $B_{\rho_M}(0_x, \varepsilon) := \{ X \in T_xM | \rho_M(X,X) \}$. Recall that on every Riemannian manifold there exists a Riemannian exponential map

$$\exp_M\colon TM \supseteq D_M \to M$$

whose domain D_M is some open neighborhood of the zero-section. Each Riemannian exponential map on a smooth Riemnannian manifold is smooth.

Recall the following standard result of Riemannian geometry:

D.0.2 Lemma *Let (M, ρ) be a Riemannian manifold with exponential map $\exp_M\colon D_M \to M$ and $K \subseteq M$ some compact subset. There is $\varepsilon > 0$ and an open set $K \subseteq V$, such that the following holds*

(a) *for each $x \in V$ the map $\exp_M|_{B_\rho(0_x,\varepsilon)}^{\exp_M(B_\rho(0_x,\varepsilon))}$ is a diffeomorphism with open image in M,*
(b) *$\bigcup_{x \in V} B_\rho(0_x, \varepsilon) \subseteq D_M$ is a neighborhood of the zero section on K.*

Proof. Apply [39, Theorem 1.8.15] to each point $x \in K$. Since K is compact, this yields a finite family $x_1, x_2, \ldots, x_n \in K$ and constants $\varepsilon_1, \ldots, \varepsilon_n$ such that:

- for each $1 \leq k \leq n$ and $y \in \exp_M(B_{\rho_M}(0_{x_k}, \varepsilon_k)$ the mapping $\exp_M|_{B_{\rho_M}(0_y,\varepsilon_k)}$ is an embedding with open image,
- $K \subseteq V := \bigcup_{1 \leq k \leq n} \exp_M(B_{\rho_M}(0_{x_k}, \varepsilon_k)$ holds.

Set $\varepsilon := \min\{ \varepsilon_1, \ldots, \varepsilon_n \}$. The pair (ε, V) satisfy the assertion of the Lemma since $\bigcup_{x \in V} B_\rho(0_x, \varepsilon)$ is a neighborhood of the zero section by the proof of [39, Theorem 1.8.15]. □

As a first step we discuss Riemannian exponential maps on metric balls in euclidean space. To this end fix the metric ball $B_5(0) \subseteq \mathbb{R}^d, d \in \mathbb{N}$ with an arbitrary Riemannian metric. For the rest of this section we endow $\mathbb{R}^d$ with the maximum norm $\|\cdot\|_\infty$. We denote by $B_r(x)$ the metric ball around $x \in B_5(0)$ with respect to $\|\cdot\|_\infty$ and $r > 0$. Endow the space $\mathcal{L}(\mathbb{R}^d)$ of linear and continuous endomorphisms of $\mathbb{R}^d$ with the operator norm $\|\cdot\|_{\text{op}}$ with respect to $\|\cdot\|_\infty$.

D.0.3 Lemma *Consider $B_5(0)$ as a Riemannian manifold with arbitrary Riemannian metric. Let* $\exp\colon D \to B_5(0)$ *be the associated Riemannian exponential map. For $t > 0$ there exist $\varepsilon > \sigma_t > 0, 1 > \delta > 0$ such that*

(a) $\overline{B_4(0)} \times \overline{B_\varepsilon(0)} \subseteq D$, $\phi_x := \exp(x,\cdot)|_{B_\varepsilon(0_x)}^{\exp(x,\cdot)(B_\varepsilon(0_x))}$ *is a diffeomorphism for each $x \in \overline{B_4(0)}$.*

(b) $B_\delta(x) \subseteq \exp(x, B_\varepsilon(0))$ *for each* $x \in \overline{B_4(0)}$, $b\colon W_\delta \to B_\varepsilon(0), b(x,y) := \phi_x^{-1}(y)$ *is a smooth map on the subset* $W_\delta := \bigcup_{x\in\overline{B_4(0)}} \{ x \} \times B_\delta(x)$ *of* $B_5(0) \times \mathbb{R}^d$

(c) $\phi_x(B_{\sigma_t}(0)) \subseteq B_t(x)$ *for each* $x \in \overline{B_4(0)}$. *For $t \leq \frac{\delta}{2}$ we obtain a smooth map*

$$f\colon B_3(0) \times B_{\sigma_t}(0) \times B_{\sigma_t}(0) \to B_\varepsilon(0), f(x,y,z) \mapsto b(x, \phi_{\phi_x(y)}(z))$$

Proof. (a) The set $\overline{B_4(0)} \times \{ 0 \}$ is a compact subset of D. Lemma D.0.2 yields a neighborhood $\overline{B_4(0)} \times \{ 0 \} \subseteq W \subseteq D$, such that $\exp(x,\cdot)$ restricts to is a diffeomorphism on $W \cap T_xM$ for $x \in \pi_{TB_5(0)}(W)$. An application of Wallace Theorem [20, 3.2.10]) yields $\varepsilon > 0$ such that $\overline{B_4(0)} \times \overline{B_\varepsilon(0)} \subseteq W$ holds.

(b) We have $\exp'(x, 0, \cdot) = \mathrm{id}_{\mathbb{R}^d}$ for each x in the compact set $\overline{B_4(0)}$ (cf. [39, Proof of Theorem 1.6.12]). Apply the parameter dependent Inverse Function Theorem [28, Theorem 5.13] to the exponential map on $\overline{B_4(0)} \times B_\varepsilon(0)$. By compactness of $\overline{B_4(0)}$ this yields some $\delta > 0$ which satisfies the assertion of (b).

(c) By uniform continuity of exp on $\overline{B_4(0)} \times \overline{B_\varepsilon(0)}$, we may choose σ_t with the desired properties. If $t \leq \frac{\delta}{2}$ holds, we obtain $\phi_{\phi_x(y)}(z) \in B_\delta(x)$ for each $(x,y,z) \in B_3(0) \times B_{\sigma_t}(0) \times B_{\sigma_t}(0)$. The assertion now follows from (b).

□

D.0.4 Lemma *Consider $B_5(0)$ as Riemannian manifold with arbitrary Riemannian metric and exponential map* exp. *Let $\rho > 0$ and ε, δ be as in Lemma D.0.3. There exists an open C^1-neighborhood $\mathcal{N}$ of the zero map in $C^\infty(B_5(0), \mathbb{R}^d)$ such that $\xi \in \mathcal{N}$ satisfies*

(a) $(\mathrm{id}_{B_5(0)}, \xi)(\overline{B_3(0)}) \subseteq \overline{B_3(0)} \times B_\varepsilon(0) \subseteq D$ *and the following estimate holds for each* $x \in B_3(0)$ $\|\exp(x, \xi(x)) - x\|_\infty \leq \min\left\{ \frac{1}{8}, \frac{\delta}{2} \right\}$,

(b) *the map* $F_\xi := \exp \circ (\mathrm{id}_{B_3(0)}, \xi|_{B_3(0)})$ *is an embedding,*

(c) *for $y \in B_3(0)$ the following estimates are available:*

$$B_{\frac{3}{4}s}(F_\xi(y)) \subseteq F_\xi(B_s(y)) \subseteq B_{\frac{5}{4}s}(F_\xi(y)), \qquad s \in]0, 3 - \|y\|] \tag{D.0.2}$$

$$B_{\frac{6s-1}{8}}(0) \subseteq F_\xi(B_s(0)) \subseteq B_{\frac{10s+1}{8}}(0), \qquad s \in]0, 3] \tag{D.0.3}$$

$$B_{\frac{8r+1}{10}}(0) \subseteq F_\xi^{-1}(B_r(0)) \subseteq B_{\frac{8r+1}{6}}(0), \qquad r \in \left]0, 2 + \tfrac{1}{8}\right] \tag{D.0.4}$$

(d) *there is a map $\xi^* \in C^\infty(\mathrm{Im}\, F_\xi, \mathbb{R}^d)$, such that $(F_\xi)^{-1} = \exp \circ (\mathrm{id}_{\mathrm{Im}\, F_\xi}, \xi^*)$ is satisfied,*

(e) $\|\xi^*\|_{\overline{B_2(0)},1} < \rho$ *holds for each $\xi \in \mathcal{N}$ and if $\xi \equiv 0$ then $\xi^* \equiv 0$,*

(f) *the map*

$$I\colon \mathcal{N} \to C^\infty(B_2(0), \mathbb{R}^d), \xi \mapsto \xi^*|_{B_2(0)}$$

is smooth.

Proof. We need preparatory estimates to control the derivatives of all relevant maps.
Since ε, δ were chosen as in Lemma D.0.3 we may consider the smooth map

$$a\colon B_4(0) \times B_\delta(0) \to B_\varepsilon(0), a(x,y) := b(x, x+y) = \phi_x^{-1}(x+y).$$

Since $\exp(x,0) = x$ holds we derive $a(x,0) = 0$ for each $x \in B_4(0)$. Thus $d_1a(x,0;\cdot) = 0$ holds for all $x \in B_4(0)$. The set $\overline{B_3(0)} \times \{0\} \subseteq a^{-1}(B_\rho(0))$ is compact, whence Wallace Lemma [20, 3.2.10] allows us to choose $0 < t \leq \min\left\{\frac{1}{8}, \frac{\delta}{2}\right\}$ with

$$a(\overline{B_3(0)} \times B_t(0)) \subseteq B_\rho(0) \tag{D.0.5}$$

$$\forall (x,y) \in \overline{B_3(0) \times B_t(0)} \quad \|d_1a(x,y;\cdot)\|_{\mathrm{op}} < \frac{\rho}{2}. \tag{D.0.6}$$

Set $m := \sup\left\{ \|d_2a(x,y;\cdot)\|_{\mathrm{op}} \middle| x \in \overline{B_3(0)}, y \in \overline{B_t(0)} \right\} < \infty$. It is well known that the invertible matrices form an open subset $\mathcal{L}(\mathbb{R}^d)^\times$ of $\mathcal{L}(\mathbb{R}^d)$ and inversion is continuous on this set (cf. [28, Proposition 1.33]). Hence there is $0 < \gamma < \frac{1}{4}$ such that for $A \in \mathcal{L}(\mathbb{R}^d)$ with $\|A - \mathrm{id}_{\mathbb{R}^d}\|_{\mathrm{op}} < \gamma$, (i.e. $A \in \mathcal{L}(\mathbb{R}^d)^\times$), we have $\left\|A^{-1} - \mathrm{id}_{\mathbb{R}^d}\right\|_{\mathrm{op}} < \frac{\rho}{2\cdot(m+1)}$.
By Lemma D.0.3 we may choose $\sigma_t > 0$ with respect to ε, δ, such that: $\varepsilon > \sigma_t$ and $\phi_x(B_{\sigma_t}(0)) \subseteq B_t(x) \subseteq B_{\min\left\{\frac{1}{8}, \frac{\delta}{2}\right\}}(x)$ holds for each $x \in \overline{B_4(0)}$. We obtain an open neighborhood of the zero-map $\lfloor \overline{B_3(0)}, B_{\sigma_t}(0) \rfloor \subseteq C(B_5(0), \mathbb{R}^d)_{\mathrm{c.o}}$and by construction each $\xi \in \lfloor \overline{B_3(0)}, B_{\sigma_t}(0) \rfloor$ satisfies the assertions of (a). We shrink $\lfloor \overline{B_3(0)}, B_{\sigma_t}(0) \rfloor$ to construct $\mathcal{N}$:
Consider $\xi \in \lfloor \overline{B_3(0)}, B_{\sigma_t}(0) \rfloor$ and define the smooth maps $F_\xi := \exp \circ (\mathrm{id}_{B_3(0)}, \xi|_{B_3(0)})$, $g_\xi := F_\xi - \mathrm{id}_{B_3(0)}$. Our goal is to apply a quantitative version of the Inverse Funtion Theorem for Lipschitz-continuous maps (cf. [28, Theorem 5.3]). From [22, Lemma 1.9] we deduce that the assignment $B_3(0) \to \mathcal{L}\left(\mathbb{R}^d\right), x \mapsto dg_\xi(x,\cdot)$ is well defined and continuous. Since the domain of g_ξ is convex, an estimate for $\|dg_\xi(z,\cdot)\|_{\mathrm{op}}$ will yield a Lipschitz-constant for g_ξ:

$$\begin{aligned} dg_\xi(z;\cdot) &= d(F_\xi - \mathrm{id}_{B_3(0)})(z;\cdot) = dF_\xi(z;\cdot) - \mathrm{id}_{\mathbb{R}^d}(\cdot) \\ &= \underbrace{d_1\exp(z,\xi(z);\cdot) - \mathrm{id}_{\mathbb{R}^d}(\cdot)}_{T_I(z)} + \underbrace{d_2\exp(z,\xi(z); d\xi(z;\cdot))}_{T_{II}(z)}, \quad z \in B_3(0) \end{aligned}$$

The map $F\colon B_4(0) \times B_\varepsilon(0) \to \mathcal{L}(\mathbb{R}^d), (z,w) \mapsto d_1\exp(z,w;\cdot) - \mathrm{id}_{\mathbb{R}^d}(\cdot)$ is continuous by [28, Lemma 3.13] with $F(x,0) = 0$ for $x \in \overline{B_3(0)}$. Let $\mathrm{pr}_2\colon B_5(0) \times \mathbb{R}^d \to \mathbb{R}^d$ be the canonical projection. Then $W_1 := \lfloor \overline{B_3(0)}, \mathrm{pr}_2(F^{-1}(B_{\frac{\gamma}{2}}^{\|\cdot\|_{\mathrm{op}}}(0))) \rfloor \subseteq C(B_5(0), \mathbb{R}^d)_{\mathrm{c.o.}}$ is an open neighborhood of the zero-map. For each $\xi \in \lfloor \overline{B_3(0)}, B_{\sigma_t}(0) \rfloor \cap W_1$ and $x \in \overline{B_3(0)}$ we derive $\|T_I(x)\|_{\mathrm{op}} \leq \frac{\gamma}{2} \leq \frac{1}{8}$.
Since $\overline{B_3(0)} \times \overline{B_\varepsilon(0)}$ is compact, there is an upper bound $\|d_2\exp(x,y;\cdot)\|_{\mathrm{op}} \leq C < \infty$. For each $\xi \in \lfloor \overline{B_3(0)}, B_{\sigma_t}(0) \rfloor \cap W_1$ and $x \in \overline{B_3(0)}$ we obtain the estimate $\|T_{II}(x)\|_{\mathrm{op}} \leq C\,\|d\xi(x;\cdot)\|_{\mathrm{op}}$.
The topology on $C^\infty(B_5(0), \mathbb{R}^d)$ is initial with respect to the family of mappings $d^{(k)}, k \in \mathbb{N}_0$ by Definition C.2.2, Thus we obtain an open C^1-neighborhood of the zero-map in $C^\infty(B_5(0), \mathbb{R}^d)$ via

$$W_2 := \left\{ \xi \in C^\infty(B_5(0), \mathbb{R}^d) \middle| d^{(1)}\xi \in \lfloor \overline{B_3(0)} \times \overline{B_1(0)}, B_{\frac{\gamma}{2C}}(0) \rfloor \right\}.$$

Define the C^1-neighborhood $\mathcal{N}$ as follows $\mathcal{N} := \lfloor \overline{B_3(0)}, B_{\sigma_t}(0) \rfloor \cap W_1 \cap W_2$. For each $\xi \in \mathcal{N}$ the construction shows $\mathrm{Lip}(g_\xi) = \sup_{\|z\|_\infty \leq 3} \|dg_\xi(z;\cdot)\|_{\mathrm{op}} \leq \gamma \leq \frac{1}{4}$.

Since $\operatorname{Lip}(g_\xi) < 1 = \frac{1}{\|\mathrm{id}_{\mathbb{R}^d}\|_{\mathrm{op}}}$ holds, the Lipschitz Inverse Function Theorem [28, Theorem 5.3] yields: For $\xi \in \mathcal{N}$ the map F_ξ is a homeomorphism onto its image and (D.0.2) is satisfied. Specialising (D.0.2) to $y = 0$ together with (a) yields (D.0.3). Apply F_ξ^{-1} to (D.0.3) to obtain (D.0.4). We claim that F_ξ is an embedding. If this were true (b) holds. To prove the claim note that for each $z \in B_3(0)$ one has $\frac{1}{4} \geq \|dg_\xi(z;\cdot)\|_{\mathrm{op}} = \|dF_\xi(z;\cdot) - \mathrm{id}_{\mathbb{R}^d}(\cdot)\|_{\mathrm{op}}$. Hence $dF_\xi(z;\cdot)$ is in $\mathcal{L}\left(\mathbb{R}^d\right)^\times$ for each $z \in B_3(0)$. The Inverse Function Theorem (see [43, I,4 Theorem 5.2]) implies that F_ξ is a local diffeomorphism and since it is already a homeomorphism onto its image, F_ξ is a smooth embedding. We are left to prove the assertions (d)-(f). To this end observe that by (c) the image of F_ξ satisfies $B_{2+\frac{1}{8}}(0) \subseteq \operatorname{Im} F_\xi \subseteq B_4(0)$. Choose $x \in \operatorname{Im} F_\xi$ and denote by $y := F_\xi^{-1}(x) \in B_3(0)$. By construction of $\mathcal{N}$, we have $\xi(y) \in B_{\sigma_t}(0)$, whence

$$x = F_\xi(y) = \phi_y(\xi(y)) \in B_t(y) \subseteq B_{\frac{\delta}{2}}(y) \tag{D.0.7}$$

and thus $y \in B_t(x)$ holds. We may thus define $\xi^*(x) := b(x, F_\xi^{-1}(x))$. This assignment is well defined for each $x \in F_\xi$ and Lemma D.0.3 (b) shows that $\overline{B_2(0)} \subseteq \operatorname{Im} F_\xi$ holds and $\xi^* \colon \operatorname{Im} F_\xi \to \mathbb{R}^d$ is smooth with $\operatorname{Im} \xi^* \subseteq B_\varepsilon(0)$. From the estimates above we deduce that $F_{\xi^*} := \exp \circ (\mathrm{id}_{\operatorname{Im} F_\xi}, \xi^*)$ is well defined. A computation with $z \in B_3(0)$ then shows

$$F_{\xi^*} \circ F_\xi(z) = \exp(F_\xi(z), \xi^*(F_\xi(z))) = (\phi_{F_\xi(z)}(\xi^* F_\xi(z)) = \phi_{F_\xi(z)}(\phi^{-1}_{F_\xi(z)} F_\xi^{-1}(F_\xi(z))) = z$$

Hence (d) holds. Notice that by construction $\xi^*(x) = a(x, (F_\xi)^{-1}(x) - x)$ holds for $x \in \operatorname{Im} F_\xi$. In particular if $\xi \equiv 0$, then $F_\xi = \mathrm{id}_{B_5(0)}$, whence $\xi^*(x) = a(x, F_\xi^{-1}(x) - x) = a(x, 0) = 0$ holds. To obtain the estimate for (e), we computes the derivative:

$$d\xi^*(x;\cdot) = d_1 a(x, (F_\xi)^{-1}(x) - x; \cdot) + d_2 a(x, (F_\xi)^{-1}(x) - x; d(F_\xi^{-1})(x;\cdot) - \mathrm{id}_{\mathbb{R}^d}(\cdot)). \tag{D.0.8}$$

By construction we have $d(F_\xi^{-1})(x;\cdot) = (dF_\xi(y;\cdot))^{-1})$ with $y := F_\xi^{-1}(x)$. By construction of $\mathcal{N}$, $\|dF_\xi(y,\cdot) - \mathrm{id}_{\mathbb{R}^d}\|_{\mathrm{op}} \leq \gamma$ and we derive $\left\|(dF_\xi(y;\cdot))^{-1} - \mathrm{id}_{\mathbb{R}^d}(\cdot)\right\|_{\mathrm{op}} < \frac{\rho}{2\cdot(m+1)}$.
Since $d(F_\xi^{-1})(x;\cdot) - \mathrm{id}_{\mathbb{R}^d} \in B_t(0)$ by (D.0.7), the operator norm of the second summand in (D.0.8) is smaller than $m \cdot \frac{\rho}{2(m+1)} < \frac{\rho}{2}$. Likewise a combination of (D.0.7) and (D.0.6) yields that the operator norm of the first summand is less than $\frac{\rho}{2}$. Summing uo $\|d\xi^*(x;\cdot\|_{\mathrm{op}} < \rho$ holds for each $x \in \operatorname{Im} \overline{B_2(0)}$. As the operator norm an the open set $\overline{B_2(0)}$ were constructed with respect to $\|\cdot\|_\infty$, we derive $\sup_{|\alpha|=1} \|\partial^\alpha \xi^*\|_{\overline{B_2(0)},0} \leq \sup_{x \in \overline{B_2(0)}} \|d\xi^*(x;\cdot)\|_{\mathrm{op}} < \rho$. On the other hand by (D.0.7) and (D.0.5) the estimate $\|\xi^*(x)\|_\infty = \|a(x, F_\xi^{-1}(x) - x)\|_\infty < \rho$ follows. In conclusion $\|\xi^*\|_{\overline{B_2(0)},1} < \rho$ and thus (e) holds.
Recall that $\xi^*(x) = a(x, (F_\xi^{-1}|_{B_{2+\frac{1}{8}}(0)} - \mathrm{id}_{B_{2+\frac{1}{8}}(0)})(x))$ holds for $x \in B_{2+\frac{1}{8}}(0) \subseteq \operatorname{Im} F_\xi$ (cf. (D.0.4)). By construction of $\mathcal{N}$ we obtain $F_\xi^{-1}|_{B_{2+\frac{1}{8}}(0)} - \mathrm{id}_{B_{2+\frac{1}{8}}(0)} \in \lfloor \overline{B_2(0)}, B_\delta(0) \rfloor_\infty \subseteq C^\infty(B_{2+\frac{1}{8}}(0), \mathbb{R}^d)$. Let a_* be the map $a_* \colon \lfloor \overline{B_2(0)}, B_\delta(0) \rfloor_\infty \to C^\infty(B_2(0), \mathbb{R}^d)$ defined via $a_*(\gamma)(x) := a(x, \gamma(x))$. This map is smooth by [25, Proposition 4.23 (a)] and since $C^\infty(B_{2+\frac{1}{8}}(0), \mathbb{R}^d)$ is a topological vector space, $\alpha \colon C^\infty(B_{2+\frac{1}{8}}(0), \mathbb{R}^d) \to C^\infty(B_{2+\frac{1}{8}}(0), \mathbb{R}^d), f \mapsto f - \mathrm{id}_{B_{2+\frac{1}{8}}(0)}$ is smooth. We claim that

$$h \colon \mathcal{N} \to C^\infty(B_{2+\frac{1}{8}}(0), \mathbb{R}^d), \xi \mapsto F_\xi^{-1}|_{B_{2+\frac{1}{8}}(0)}$$

is smooth. If this were true the assertion of (f) follows, since $I = a_* \circ \alpha \circ h$ holds. Remark C.2.3 (a) implies that the space $C^\infty(B_5(0), \mathbb{R}^d)$ is metrizable. Hence by [25, Proposition E.3] h is a smooth map, if and only if the map $h \circ c$ is smooth for each smooth curve $c\colon \mathbb{R} \to \mathcal{N}$. By the Exponential law [2, Theorem 3.28], the map $h \circ c\colon \mathbb{R} \to C^\infty(B_{2+\frac{1}{8}}(0), \mathbb{R}^d)$ will be smooth if and only if $(h \circ c)^\wedge\colon \mathbb{R} \times B_{2+\frac{1}{8}}(0) \to \mathbb{R}^d, (t,x) \mapsto h(c(t))(x)$ is smooth. To prove this, we adapt an argument from [42, p. 455]: Consider the map

$$H\colon \mathbb{R} \times B_{2+\frac{1}{8}}(0) \times B_3(0) \to \mathbb{R}^d, (t,x,y) \mapsto \exp(y, c^\wedge(t,y)) - x = F_{c(t)}(y) - x$$

which is well defined by construction of $\mathcal{N}$. Furthermore H is smooth, as $c^\wedge\colon \mathbb{R} \times B_5(0) \to \mathbb{R}^d$ is smooth by [2, Theorem 3.28]. Observe that since $F_{c(t)} \circ h(c(t))(x) = x$ holds for each $t \in \mathbb{R}, x \in B_{2+\frac{1}{8}}(0)$ we obtain the identity $H(t,x,(h \circ c)^\wedge(t,x)) = 0$ (note that by (c) we may compose H and $(h \circ c)^\wedge$). A computation yields the following estimate for the derivative of H:

$$\begin{aligned}
\|d_3 H(t,x,y;\cdot) - \mathrm{id}_{\mathbb{R}^d}(\cdot)\|_{\mathrm{op}} &= \|d_1 \exp(y, c^\wedge(t,y);\cdot) + d_2 \exp(y, c^\wedge(t,y); d_2 c^\wedge(t,y;\cdot)) - \mathrm{id}_{\mathbb{R}^d}(\cdot)\|_{\mathrm{op}} \\
&\leq \|d_1 \exp(y, c^\wedge(t,y);\cdot) - \mathrm{id}_{\mathbb{R}^d}\|_{\mathrm{op}} + C \cdot \|d_2 c^\wedge(t,y;\cdot)\|_{\mathrm{op}} \\
&\leq \frac{\gamma}{2} + \frac{\gamma}{2} \leq \frac{1}{8} + \frac{1}{8} < 1
\end{aligned}$$

Here we used the estimates for T_I, T_{II} and $d_2 c^\wedge$ obtained above. An estimate for $c^\wedge$ is available since $c(t) \in \mathcal{N}$ holds for $t \in \mathbb{R}$. We deduces that $d_3 H(t,x,y;\cdot)$ is invertible for each $(t,x,y) \in \mathbb{R} \times B_{2\frac{1}{8}}(0) \times B_3(0)$. Furthermore for fixed $(t,x) \in \mathbb{R} \times B_{2\frac{1}{8}}(0)$ the map $H(t,x,\cdot) = F_{c(t)}(\cdot) - x$ is injective on $B_3(0)$. Using the injectivity, we deduce with the Implicit Function Theorem [28, Theorem 5.2] that $(h \circ c)^\wedge$ is smooth. In conclusion (f) holds. □

D.0.5 Lemma ([43, II.3 Theorem 3.3]) *Let M be a finite dimensional paracompact Hausdorff manifold of dimension d. Given an open covering $\mathcal{O}$ of M there exists a locally finite manifold atlas $\mathcal{V}(\mathcal{O}) := \{(V_{5,k}, \kappa_k)\}_{k \in I}$ with the following properties:*

(a) *the covering $\mathcal{V}(\mathcal{O})$ is subordinate to $\mathcal{O}$ and each chart in $\mathcal{V}(\mathcal{O})$ is precompact,*
(b) *for each $k \in I$ one has $\kappa_k(V_{5,k}) = B_5(0) \subseteq \mathbb{R}^d$,*
(c) *the open sets $V_{\tau,k} := \kappa_k^{-1}(B_\tau(0))$ cover M for $\tau \in [1,5]$,*
(d) *if M is σ-compact, we may choose a countable atlas with properties (a) - (c).*

Proof. The manifold M is locally compact and paracompact. Apply [20, Lemma 5.1.6] together with local compactness of M to obtain a refinement $\mathcal{O}'$ of the covering $\mathcal{O}$, such that the closure of each of the open sets in $\mathcal{O}'$ is compact and contained in some open set in $\mathcal{O}$. By Proposition 2.4.2 each component of M is second countable and thus we may apply [43, II.3 Theorem 3.3] to obtain a (countable) manifold atlas subordinate to $\mathcal{O}'$ for each component. Thus the closure of any chart domain in this atlas is compact as closed subset of a compact set. Taking the union of the atlases for the components, we obtain an atlas $\mathcal{V}(\mathcal{O})$ for M with the desired properties. If M is σ-compact, there are at most countably many components, whence the above construction yields a countable atlas. □

We shall combine our observations to construct special neighborhoods of the zero-section in $\mathfrak{X}(M)$ for a paracompact Riemannian manifold (M, ρ_M). Consider some atlas $\{ (V_{5,k}, \kappa_k) | k \in I \}$ as in Lemma D.0.5 on (M, ρ_M). For each chart $(V_{5,k}, \kappa_k)$ we define the pullback Riemannian metric ρ_k on $B_5(0)$ with respect to κ_k^{-1}. Then κ_k^{-1} becomes a Riemannian embedding, i.e.

$$T\kappa_k^{-1}(B_{\rho_k}(0_{\kappa_k(x)}, r)) = B_\rho(0_x, r) \quad r > 0 \tag{D.0.9}$$

holds for $x \in V_{5,k}$. The Riemannian exponential map $\exp_k$ associated to the Riemannian pullback metric ρ_k thus satisfies $T\kappa_k^{-1}(\operatorname{dom} \exp_k) \subseteq \operatorname{dom} \exp_M$ and

$$\exp_M T\kappa_k^{-1}|_{\operatorname{dom} \exp_k} = \kappa_k^{-1} \exp_k \tag{D.0.10}$$

For the rest of this section we endow each manifold chart with the Pullback Riemannian metric induced with respect to the chart map. Whenever the constructions require a Riemannian metric on a chart, we use the induced metric without further mentioning it.

D.0.6 Lemma *Let (M, ρ_M) be a d-dimensional Hausdorff paracompact Riemannian manifold with Riemannian exponential map $\exp_M$ and some open covering $\mathcal{O}$ of M. Choose via Lemma D.0.5 an atlas $\mathcal{V}(\mathcal{O}) := \{ (V_{5,k}, \kappa_k | k \in I \}$ with respect to $\mathcal{O}$. For each $k \in I$ there is $\nu_k > 0$ such that*

(a) *for each $y \in \overline{V_{4,k}}$ the map $\exp_M$ is injective on $N_y := \bigcup_{n \in I_y} T\kappa_n^{-1}(\{ \kappa_n(y) \} \times B_{\nu_n}(0)) \subseteq T_y M$, where the index set is defined as $I_y := \left\{ k \in I \middle| y \in \overline{V_{4,k}} \right\}$.*
(b) *We have $T\kappa_n(N_y) \subseteq \operatorname{dom} \exp_n$, $\exp_n |_{T\kappa_n(N_y)}$ is an embedding with open image and the identity $\exp_n T\kappa_n|_{N_y} = \kappa_n \exp_M |_{N_y}$ is satisfied for each $n \in I_y$.*

If I is finite, we may choose $\nu > 0$ such that (a), (b) holds for each $k \in I$ with respect to ν. Furthermore if one applies Lemma D.0.4 to each $k \in I$ with the additional requirement $\varepsilon_k < \nu$, then one obtains a family of open C^1-zero-neighborhoods $\mathcal{N}_k \subseteq C^\infty(V_{5,k}, \mathbb{R}^d)$ such that for $X \in \theta_{\kappa_k}^{-1}(\mathcal{N}_k)$

(c) the map $\exp_M \circ X|_{\overline{V_{3,k}}}$ is well defined, with $\operatorname{Im} \exp_M \circ X|_{\overline{V_{3,k}}} \subseteq V_{5,k}$
(d) The following estimates are available: $\exp_M \circ X(\overline{V_{\frac{5}{4},k}}) \subseteq V_{2,k}$, $V_{\frac{5}{4},k} \subseteq \exp_M \circ X(V_{2,k}) \subseteq V_{3,k}$ and $B_4(0) \times B_\nu(0) \subseteq \operatorname{dom} \exp_k$ is satisfied for $k \in I$.
(e) the map $F_X^k := \exp_M \circ X|_{V_{3,k}}$ is a smooth embedding,
(f) for each $x \in \overline{V_{3,k}}$ we have $X_{\kappa_k}(x) \in B_\nu(0)$.

Proof. For each $k \in I$ Lemma D.0.3 allows us to choose $\nu_k' > 0$ such that $\exp_k(x, \cdot)$ restricts to an embedding with open image on $\overline{B_{\nu_k'}(0)}$ for each $x \in \overline{B_4(0)}$. Since $\overline{V_{4,k}}$ is compact, and the covering $\mathcal{V}$ is locally finite, there is a finite subset $F_k \subseteq I$, such that $V_{5,i} \cap \overline{V_{4,k}} \neq \emptyset$ if and only if $i \in F_k$. By compactness of $\overline{V_{4,k}} \cap \overline{V_{4,j}}, j \in F_k$, there is some $\nu_k > 0$ such that for each $j \in F_k$, one has $T\kappa_j \circ \kappa_k^{-1}(\{ \kappa_k(x) \} \times B_{\nu_k}(0)) \subseteq \{ \kappa_j(x) \} \times B_{\nu_j'}(0)$ for $x \in \overline{V_{4,k}} \cap \overline{V_{4,j}}$. The choice of ν_k' together with (D.0.10) shows that the open sets N_x induced by the family $\{ \nu_k | k \in I \}$ satisfy the assertion of (a). Since $T\kappa_n(N_x) \subseteq \{ \kappa_n(x) \} \times B_{\nu_n'}(0)$ holds for each $n \in I_x$ by construction, the set $T\kappa_n(N_x)$ is contained in the domain of $\exp_n$ for each $n \in I_x$. Hence (D.0.10) yields $\exp_M |_{N_x} = \exp_M T\kappa_k^{-1}|_{\operatorname{dom} \exp_k} T\kappa_k|_{N_x} = \kappa_k^{-1} \exp_k T\kappa_k|_{N_x}$. We deduce that (b) must hold.

If I is finite, choose $\nu := \min \{ \nu_k | k \in I \}$. We are left to construct the open sets $\mathcal{N}_k$. Fix $k \in I$ and consider the chart $(V_{5,k}, \kappa_k)$. Reviewing Lemma D.0.4 the construction of $\mathcal{N}'_k \subseteq C^\infty(B_5(0), \mathbb{R}^d)$ may be carried out using arbitrarily small ε, since by hypothesis ε must have the same properties as in Lemma D.0.4, where it may be chose arbitrarily small. The map κ_k is a diffeomorphism, whence the pullback $C^\infty(\kappa_k, \mathbb{R}^d) \colon C^\infty(B_5(0), \mathbb{R}^d) \to C^\infty(V_{5,k}, \mathbb{R}^d), f \mapsto f \circ \kappa_k$ is linear bijective and continuous by a combination of [25, Lemma 4.11] and [5, Proposition 7.4]. Define the open C^1-neighborhood $\mathcal{N}_k := C^\infty(\kappa_k, \mathbb{R}^d)^{-1}(\mathcal{N}'_k) \subseteq C^\infty(V_{5,k}, \mathbb{R}^d)$. The Riemannian exponential map $\exp_k$ is related to $\exp_M$ via (D.0.10) and the identity in (b). Hence the properties obtained via Lemma D.0.4 for vector fields with $X_{\kappa_k} \in \mathcal{N}_k$ imply (c) - (f). $\square$

In the setting of Lemma D.0.6 consider a compact subset $K \subseteq M$. As $\mathcal{V}(\mathcal{O})$ is locally finite, there is a finite subset $\mathcal{F}_5(K) := \left\{ (V_{5,k}, \kappa_{k_j}) \middle| 1 \leq j \leq N \right\}$ of $\mathcal{V}(\mathcal{O})$ such that $V_{5,k} \cap K \neq \emptyset$ holds if and only if $(V_{5,k}, \kappa_k) \in \mathcal{F}_5(K)$. Notice that $\mathcal{F}_5(K)$ induces a family of open neighborhoods of K via

$$K \subseteq \Omega_{r,K} := \bigcup_{l=1}^{N} V_{r,k_l}, \quad r \in [1,5]$$

The set $\mathcal{F}_5(K)$ is finite, whence the set $K_5 := \bigcup_{i \in \mathcal{F}_5(K)} \overline{V_{5,i}}$ is compact. Again we define a finite subset $\mathcal{F}_5(K_5) := \{ (V_{5,n}, \kappa_n) | n \in I, V_{5,n} \cap K_5 \neq \emptyset \}$ of $\mathcal{V}(\mathcal{O})$ as the family of charts, which intersect the compact set K_5. As above one defines open neighborhoods Ω_{r,K_5} of K_5 for $r \in [1,5]$.

D.0.7 Lemma *Let $K \subseteq M$ be a compact set and $\mathcal{F}_5(K) = \{ (V_{5,k}, \kappa_k) | 1 \leq k \leq N \}$ as above. Construct for each $1 \leq k \leq N$ a C^1-zero-neighborhood $\mathcal{N}_k \subseteq C^\infty(V_{5,k}, \mathbb{R}^d)$ as in Lemma D.0.6 (c)-(f). Furthermore consider the continuous maps $\theta_{\kappa_k}^{\Omega_{5,K}} \colon \mathfrak{X}(\Omega_{5,K}) \to C^\infty(V_{5,k}, \mathbb{R}^d), X \mapsto X_{\kappa_k}$. There are open C^1-zero-neighborhoods $\mathcal{M}_k \subseteq \mathcal{N}_k$ with $E_{5,K} := \bigcap_{k=1}^N (\theta_k^{\Omega_{5,K}})^{-1}(\mathcal{M}_k) \subseteq \mathfrak{X}(\Omega_{5,K})$ and $E := (\operatorname{res}^M_{\Omega_{5,K}})^{-1}(E_{5,K}) \subseteq \mathfrak{X}(M)$, such that $F_X := \exp_M \circ X|_{\Omega_{2,K}}$ is a smooth embedding for each $X \in E$. In addition $F_X(\overline{\Omega_{1,K}}) \subseteq \Omega_{2,K}$ holds.*

Proof. The proof is a variation of [36, 2. Theorem 1.4]. $K_k :=$ By Lemma D.0.6 for each $X \in \theta_{\kappa_k}^{-1}(\mathcal{N}_k)$ the map $\exp_M \circ X|_{\overline{V_{2,k}}}$ is well-defined and its image is contained in $V_{3,k}$ for each $(V_{5,k}, \kappa_k) \in \mathcal{F}_5(K)$. The manifold M is locally compact, whence a regular space. Note that each of the sets $\overline{V_{2,k}}$ is compact and $M \setminus V_{3,k}$ is a closed set. Therefore for each $(V_{5,k}, \kappa_k) \in \mathcal{F}_5(K)$ there are disjoint open sets $A_k, B_k \subseteq M$ by [20, Theorem 3.1.6.], such that $\overline{V_{2,k}} \subseteq A_k$ and $M \setminus V_{3,k} \subseteq B_k$ holds.
Claim: There is a family of neighborhoods of the zero-map $\mathcal{M}_k \subseteq \mathcal{N}_k$, $1 \leq k \leq N$, such that for $X \in E_{5,K}$ the following holds: $F_X(\overline{V_{2,k}} \cap \Omega_{2,K}) \subseteq A_k$ and $F_X(\Omega_{2,K} \setminus V_{3,k}) \subseteq B_k$ for each $1 \leq k \leq N$.
If this were true, then the proof may be completed as follows:
Let X be contained in $E_{5,k}$. Observe that the construction of $E_{5,k}$ implies that for each $1 \leq k \leq N$ the map $F_X|_{V_{3,k} \cap \Omega_{2,K}} = F_X^k|_{V_{3,k} \cap \Omega_{2,k}}$ is an embedding by Lemma D.0.6 (e). Consider distinct $x, y \in \Omega_{2,K}$ and choose $1 \leq k \leq N$ with $x \in \overline{V_2, k}$. If $y \in V_{3,k}$ we must have $F_X(x) \neq F_X(y)$ since the map is an embedding on $V_{3,k}$. On the other hand, if $y \in \Omega_{2,K} \setminus V_{3,k} \subseteq M \setminus V_{3,k}$ holds, by the above $F_X(x) \in F_X(\overline{V_{2,k}}) \subseteq A_k$ and $F_X(y) \in F_X(\Omega_{2,K} \setminus V_{3,k}) \subseteq B_k$. Since A_k and B_k are disjoint, again $F_X(x) \neq F_X(y)$ follows, whence $F_X|_{\Omega_{2,K}}$ must be injective. Thus each $X \in E$ yields an injective

local diffeomorphism $\exp_M \circ X|_{\Omega_{2,K}}$, i.e. $\exp_M \circ X|_{\Omega_{2,K}}$ is an embedding. Furthermore F_X maps $\overline{V_{1,k}}$ into $V_{2,k}$ by Lemma D.0.6 (d). Hence the definition of $\Omega_{1,K}$ and $\Omega_{2,K}$ yield $F_X(\overline{\Omega_{1,K}}) \subseteq \Omega_{2,K}$.

Proof of the claim: For $k \neq j$ we obtain a sets

$$K_{kj} := \kappa_k(\overline{V_{2,k}} \cap M \setminus V_{3,j}) \subseteq \overline{B_2(0)} \text{ and } B_{kj} := T\kappa_k(TV_{5,k} \cap \exp_M^{-1}(B_j \cap V_{3,k}))$$

By construction each set $K_{kj} \subseteq B_5(0)$ is compact and each set B_{kj} is an open subset of $TB_5(0)$. Furthermore set $A_{kk} := T\kappa_k(TV_{5,k} \cap \exp_M^{-1}(A_k))$, for $1 \leq k \leq N$. Recall that for the zero section $\exp_M \circ 0_M = \mathrm{id}_M$ holds. For $(k,j) \in \{\, 1 \leq k, j \leq N | k \neq j \,\}$ this yields the following inclusions $K_{kj} \times \{\, 0 \,\} \subseteq B_{kj}$ and $K_{kk} \times \{\, 0 \,\} \subseteq A_{kk}$. For each $1 \leq k \leq N$ we obtain an open neighborhood

$$M_k := \lfloor \overline{B_2(0)}, A_{kk} \rfloor \cap \bigcap_{\substack{j=1 \\ j \neq k}}^{N} \lfloor K_{kj}, B_{kj} \rfloor \subseteq C^\infty(B_5(0), \mathbb{R}^d)$$

of the zero-map. We obtain a C^1-open set $\mathcal{M}_k := C^\infty(\kappa_k, \mathbb{R}^d)^{-1}(M_k) \cap \mathcal{N}_k \subseteq C^\infty(V_{5,k}, \mathbb{R}^d)$. By construction each vector field $X \in E_{5,K}$ (defined as in the statement of the Lemma) may be composed on $\Omega_{3,K}$ with $\exp_M$. With the identities (D.0.10) and Lemma D.0.6 (b) the mapping F_X may be evaluated locally on $V_{2,k}$ only in the chart $(V_{5,k}, \kappa_k) \in \mathcal{F}_5(K)$. For any $X \in E_{5,K}$ we note that $X_{\kappa_k} \in C^\infty(\kappa_k, \mathbb{R}^d)^{-1}(\lfloor \overline{B_2(0)}, A_{kk} \rfloor)$ holds. By definition of A_{kk} a computation in $(V_{5,k}, \kappa_k)$ therefore yields: $F_X(\overline{V_{2,k}}) \subseteq A_k$. Furthermore each element $y \in \Omega_{2,K} \setminus V_{3,k}$ is contained in $\overline{V_{2,n}}$ for some $1 \leq n \leq N$. Thus $\kappa_n(y)$ is contained in K_{nk} by construction. Furthermore $X_{\kappa_n} \in C^\infty(\kappa_n, \mathbb{R}^d)^{-1}(\lfloor K_{nk}, B_{nk} \rfloor)$ holds. By definition of B_{nk}, a computation in the chart $(V_{5,n}, \kappa_n)$ thus yields: $F_X(y) \in B_k$. As $y \in \Omega_{2,K} \setminus V_{3,k}$ and k were chosen arbitrary, $F_X(\Omega_{2,K} \setminus V_{3,k}) \subseteq B_k$ holds for each $1 \leq k \leq N$. □

We are interested in vector field, which yield, after composition with the Riemannian exponential map, an inverse for F_X respectively the composition $F_Y \circ F_X$. In the rest of this section we construct C^1-neighborhoods of the zero-section, whose elements permit such vector fields. Furthermore the mappings sending a vector fields to the vector field which induces $F_X \circ F_Y$ respectively F_X^{-1} should be smooth on these neighborhoods. The leading idea is to construct these fields locally in a cover of charts, which will enable us to obtain them as global objects from the local data. For reasons which are explained in Section 6, we construct a neighborhood of the zero-section depending on a open C^1-neighborhood of the zero-section chosen in advance and on a positiv constant R.

D.0.8 Construction Consider the setting of Lemma D.0.7: Let $K \subseteq M$ be compact and $E_{5,K} \subseteq \mathfrak{X}(\Omega_{5,K})$ an open neighborhood of the zero-section as in Lemma D.0.7. Fix $R > 0$ and an arbitrary open C^1-neighborhood of the zero-section $P \subseteq \mathfrak{X}(\Omega_{5,K})$. By construction of the manifold atlas $\Omega_{5,K} \subseteq \Omega_{1,K_5}$ holds by Lemma D.0.5 (c). As the family $\mathcal{F}_5(K_5)$ is a manifold atlas for Ω_{5,K_5}, the topology on $\mathfrak{X}(\Omega_{1,K_5})$ is initial with respect to the family $\left\{ \theta_{\kappa_k|_{V_{1,k}}} \middle| (V_{5,k}, \kappa_k) \in \mathcal{F}_5(K_5) \right\}$ by Definition C.3.1. Thus there is a family of open C^1-neighborhoods of the zero-map $W_k \subseteq C^\infty(V_{1,k}, \mathbb{R}^d) \cong$

$C^\infty(B_1(0),\mathbb{R}^d),(V_{5,k},\kappa_k) \in \mathcal{F}_5(K_5)$ with $(\mathrm{res}^{\Omega_{1,K_5}}_{\Omega_{5,K}})^{-1}(E_{5,K}\cap P) = \bigcap_{\mathcal{F}_5(K_5)} \theta_{\kappa_k|_{V_{1,k}}}(W_k)$. Since $B_1(0) \subseteq B_5(0) = \kappa_k(V_{5,k})$ holds, Remark C.2.6 (a) implies that we may choose $\tau > 0$ such that for $f \in B^k_\tau := \left\{ f \in C^\infty(\kappa_k(V_{5,k}),\mathbb{R}^d) \middle| \|f\|_{\overline{B_1(0)},1} < \tau \right\}$ the condition $f|_{B_1(0)} \in W_k$ is satisfied. Shrinking τ if necessary, we may assume $\tau < R$. Define the open C^1-neighborhood of the zero-section $E' := \bigcap_{\mathcal{F}_5(K_5)}(\theta^{\Omega_{5,K_5}}_{\kappa_k})^{-1}C^\infty(\kappa_k^{-1},\mathbb{R}^d)(B^k_\tau) \subseteq \mathfrak{X}(\Omega_{5,K_5})$. By construction the inclusions $E' \subseteq (\mathrm{res}^{\Omega_{5,K_5}}_{\Omega_{5,K}})^{-1}(E_{5,K}\cap P)$ and $(\mathrm{res}^M_{\Omega_{5,K_5}})^{-1}(E') \subseteq E \cap (\mathrm{res}_{\Omega^M_{5,K}})^{-1}(P)$ hold.

Step 1: A vector field inducing the composition $\exp_M \circ X \circ F_Y$**:** Since the family $\mathcal{F}_5(K_5)$ is finite, we may fix a constant $\nu > 0$ as in Lemma D.0.6. Consider arbitrary $(V_{5,n},\kappa_n) \in \mathcal{F}_5(K_5)$ and shrink the C^1-open set B^n_τ: Choose $\varepsilon_n > \sigma_{\frac{\delta_n}{2}} > 0$ and $1 > \delta_n > 0$ with properties as in Lemma D.0.3, such that $\varepsilon_n < \min\{\tau,\nu\}$ holds. Set $\sigma_n := \sigma_{\frac{\delta_n}{2}}$ and $\rho_n := \min\{\nu,\tau\}$. Apply Lemma D.0.4 with the constants $\varepsilon_n,\delta_n,\rho_n$ taking the roles of ε,δ,ρ to obtain a C^1-neighborhood $\mathcal{N}_n$ of the zero-map in $C^\infty(B_5(0),\mathbb{R}^n)$. Then each $X \in C^\infty(\kappa_n^{-1},\mathbb{R}^d)(\mathcal{N}_n) \subseteq C^\infty(V_{5,n},\mathbb{R}^d)$ satisfies the assertions of Lemma D.0.6 (c)-(e) with respect to ν. By choice of the constants (cf. Lemma D.0.3), there is a smooth map $f_n\colon B_3(0)\times B_{\sigma_n}(0)\times B_{\sigma_n}(0) \to B_{\varepsilon_n}(0)$ with

$$f_n(x,0,0)=0, f_n(x,y,0)=y \text{ and } f_n(x,0,z)=z, \quad (x,y,z)\in B_3(0)\times B_{\sigma_n}(0)\times B_{\sigma_n}(0). \tag{D.0.11}$$

Hence the partial derivative satisfies $d_1 f_n(x,0,0;\cdot) = 0$, for all $x \in B_3(0)$. The continuous map $B_3(0)\times B_{\sigma_n}\times B_{\sigma_n}, (x,y,z)\mapsto \|df_n(x,y,z;\cdot)\|_{\mathrm{op}}$ is bounded on $\overline{B_2(0)}\times\overline{B_{\frac{\sigma_n}{2}}(0)}\times\overline{B_{\frac{\sigma_n}{2}}(0)}$ by some $t_n > 0$. As the partial derivative with respect to x vanishes in $\overline{B_2(0)}\times\{0\}\times\{0\}$, a compactness argument yields $0 < \mu_n < \min\left\{\nu, \frac{\sigma_n}{2}, \frac{\tau}{6dt_n}\right\}$ such that for all $(x,y,z)\in\overline{B_2(0)}\times\overline{B_{\mu_n}(0)}\times\overline{B_{\mu_n}(0)}$ the estimate $\|d_1 f_n(x,y,z;\cdot)\|_{\mathrm{op}} < \frac{\tau}{3}$holds. Define the open C^1-zero-neighborhood

$$\mathcal{H}'_n := \mathcal{N}_n \cap \left\{ f\in C^\infty(B_5(0),\mathbb{R}^d) \middle| \|f\|_{\overline{B_3(0)},1} < \mu_n \right\} \subseteq C^\infty(B_5(0),\mathbb{R}^d)$$

Since $\mu_n < \tau$ holds, we deduce $\mathcal{H}'_n \subseteq B^n_\tau$. Set $\mathcal{H}' := \bigcap_{\mathcal{F}_{5,K_5}}(\theta^{\Omega_{5,K_5}}_{\kappa_n})^{-1}C^\infty(\kappa_n,\mathbb{R}^d)^{-1}(\mathcal{H}'_n) \subseteq \mathfrak{X}(\Omega_{5,K_5})$ to obtain a C^1-neighborhood of the zero-section contained in E'.
Let ξ,η be elements of $\mathcal{H}'_n$. By Lemma D.0.4 $F_\xi(B_2(0)) \subseteq B_3(0)$ holds, whence the composition $F_\eta \circ F_\xi|_{B_2(0)}$ is well-defined. Since $\mu_n < \sigma_n$ holds, we have $F_\eta F_\xi(x) \in B_{\delta_n}(x)$ for each $x\in B_2(0)$ by Definition of $\sigma_n = \sigma_{\frac{\delta_n}{2}}$ (cf. Lemma D.0.3). Therefore for each $x \in B_2(0)$ the assignment

$$\eta\diamond\xi(x) := \phi_x^{-1}(F_\eta F_\xi(x)) = f_n(x,\xi(x),\eta(F_\xi(x))) \in B_{\varepsilon_n}(0) \subseteq B_\tau(0) \tag{D.0.12}$$

is well defined and yields a smooth map $\eta\diamond\xi\colon B_2(0)\to B_{\varepsilon_n}(0)\subseteq B_\tau(0)$. Observe that $\eta,\xi\equiv 0$ implies $\eta\diamond\xi\equiv 0$ by (D.0.11). For $(V_{5,n},\kappa_n)\in\mathcal{F}_5(K_5)$ and $X\in\mathcal{H}'$ set $X_{[n]} := X_{\kappa_n}\circ\kappa_n^{-1}$. Consider $y \in V_{3,n}$ and $X\in\mathcal{H}'$. By construction $X_{[n]}\in\mathcal{H}'_n$ holds, whence $X_{[n]}(\kappa_n(y)) \in B_{\mu_n}(0)\subseteq B_\nu(0)$. Since $\{\kappa_n(y)\}\times B_\nu(0) \subseteq T\kappa_n(N_y)$ holds, Lemma D.0.6 (b) yields for $F_{X_{[n]}}$ as in Lemma D.0.4

$$\begin{aligned}\kappa_n^{-1}F_{X_{[n]}}(\kappa_n(y)) &= \kappa_n^{-1}\exp_n(\kappa_n(y),X_{[n]}(\kappa_n(y))) = \kappa_n^{-1}\exp_n T\kappa_n\circ X(y)\\ &= \exp_M T\kappa_n^{-1}T\kappa_n\circ X(y) = \exp_M\circ X(y) = F_X(y).\end{aligned}$$

Furthermore a combination of Lemma D.0.6 (b) and (c) allows us to compute the identity

$$T\kappa_n(\exp_M|_{N_y})^{-1}\kappa_n^{-1}|_{\exp_n(T\kappa_n(N_y)} = (\exp_n|_{T\kappa_n(N_y)})^{-1}$$

for $y \in V_{3,n}$. Set $x := \kappa_n(y)$ with $y \in V_{2,n}$. Since $\varepsilon_n < \nu$ holds, we conclude $\{x\} \times B_\nu(0) \subseteq T\kappa_n(N_y)$. This yields the following identity:

$$\begin{aligned}
&(\mathrm{id}_{B_2(0)}, X_{[n]} \diamond Y_{[n]})(x) = (x, f_n(x, Y_{[n]}(x), X_{[n]}(F_{Y_{[n]}}(x)))) = (\exp_n|_{\{x\}\times B_{\varepsilon_n}(0)})^{-1} F_{X_{[n]}} F_{Y_{[n]}}(x)) \\
=&(\exp_n|_{T\kappa_n(N_y)})^{-1} F_{X_{[n]}} F_{Y_{[n]}}(x) = T\kappa_n(\exp_M|_{N_y})^{-1}\kappa_n^{-1}|_{\exp_n(T\kappa_n(N_y))} F_{X_{[n]}} F_{Y_{[n]}}(x) \\
=&T\kappa_n(\exp_M|_{N_y})^{-1}\kappa_n^{-1} F_{X_{[n]}} \circ F_{Y_{[n]}}(x))) = T\kappa_n(\exp_M|_{N_y})^{-1}\exp_M X\kappa_n^{-1} F_{Y_{[n]}}(x) \\
=&T\kappa_n(\exp_M|_{N_y})^{-1}\exp_M X \exp_M Y(y) = T\kappa_n(\exp_M|_{N_y})^{-1}\exp_M \circ X(F_Y(y))
\end{aligned} \tag{D.0.13}$$

This assignment is well-defined and smooth on each $V_{2,k}$ by (D.0.12). Hence for $X, Y \in \mathcal{H}'$ we define $X \diamond Y : \Omega_{2,K_5} \to TM, x \mapsto (\exp_M|_{N_x})^{-1}(\exp_M \circ Y \circ \exp_M \circ X)(x)$, which is an element of $\mathfrak{X}(\Omega_{2,K_5})$. The identity (D.0.13) yields $X \diamond Y \equiv 0$ for $X, Y \equiv 0$ and $(X \diamond Y)_{[n]}|_{B_2(0)} = (X_{[n]} \diamond Y_{[n]})$. From (D.0.12) we deduce

$$\left\|(X \diamond Y)_{[n]}\right\|_{\overline{B_{\frac{3}{2}}(0)},0} = \left\|X_{[n]} \diamond Y_{[n]}\right\|_{\overline{B_{\frac{3}{2}}(0)},0} < \varepsilon_n < \tau < R. \tag{D.0.14}$$

Step 2: A vector field inducing F_X^{-1}: By construction each $\mathcal{H}'_n, (V_{5,n}, \kappa_n) \in \mathcal{F}_5(K_5)$ is contained in a set $\mathcal{N}_n$ constructed via Lemma D.0.4 such that the assertion of Lemma D.0.6 (c) -(e) hold. In particular we may apply Lemma D.0.7 with $K = K_5$, open covering $\mathcal{F}_5(K_5)$ and open sets $(\mathcal{H}'_n)_{(V_{5,n},\kappa_n)\in\mathcal{F}_5(K_5)}$: For each chart in $\mathcal{F}_5(K_5)$ we obtain an open C^1-zero-neighborhood $\mathcal{H}_n \subseteq C^\infty(\kappa_n, \mathbb{R}^d)^{-1}(\mathcal{H}'_n) \subseteq C^\infty(V_{5,n}, \mathbb{R}^d)$. Then define $\mathcal{H}_R^{\Omega_{5,K_5}} := \bigcap_{(V_{5,n},\kappa_n)\in\mathcal{F}_5(K_5)} (\theta_{\kappa_n}^{\Omega_{5,K_5}})^{-1}(\mathcal{H}_n) \subseteq \mathcal{H}'$. By Lemma D.0.6 (e) for each $X \in \mathcal{H}_R^{\Omega_{5,K_5}}$ the map $\exp_M \circ X|_{\Omega_{2,K_5}}$ is a smooth embedding.

Consider $X \in \mathcal{H}_R^{\Omega_{5,K_5}}$ and $(V_{5,n}, \kappa_n) \in \mathcal{F}_5(K_5)$. By construction of $\mathcal{H}'_n$ in Step 1 we deduce with Lemma D.0.4 (c) that $B_{\frac{5}{4}}(0) \subseteq F_{X_{[n]}}(B_2(0))$ holds. We already established the identities $F_X(y) = \kappa_n^{-1} F_{X_{[n]}}(\kappa_n(y))$ and $T\kappa_n(\exp_M|_{N_y})^{-1}\kappa_n^{-1}|_{\exp_n(T\kappa_n(N_y)} = (\exp_n|_{T\kappa_n(N_y)})^{-1}$for $y \in V_{3,n}$ and $X \in \mathcal{H}_R^{\Omega_{5,K_5}}$. Furthermore Lemma D.0.4 (c)-(e) yields a map $X^*_{[n]} \in C^\infty(\operatorname{Im} F_{X_{[n]}}, \mathbb{R}^d)$ with $F_{X^*_{[n]}} = \exp_n(\mathrm{id}_{\operatorname{Im} F_{X_{[n]}}}, X^*_{[n]}) = F_{X_{[n]}}^{-1}$. This map satisfies $\left\|X^*_{[n]}\right\|_{\overline{B_2(0)},1} < \rho_n = \min\{\nu, \tau\} < R$. Hence by choice of ν we deduce $X^*_{[n]}(y) \in T\kappa_n(N_y)$ and thus $F_{X^*_{[n]}}(y) \in \exp_n(T\kappa_n(N_y))$ holds for each $y \in V_{\frac{5}{4}}$. Combining these facts we compute for $(V_{5,n}, \kappa_n) \in \mathcal{F}_5(K_5)$ and $y \in V_{\frac{5}{4}}$:

$$\begin{aligned}
T\kappa_n^{-1}(\exp_n|_{T\kappa_n(N_y)})^{-1} F_{X^*_{[n]}}(\kappa_n(y)) &= (\exp_M|_{N_y})^{-1}\kappa_n^{-1}(F_{X_{[n]}})^{-1}(\kappa_n(y)) \\
&= (\exp_M|_{N_y})^{-1}(\kappa_n^{-1} F_{X_{[n]}}\kappa_n)^{-1}(y) \\
&= (\exp_M|_{N_y})^{-1} F_X^{-1}(y) = (\exp_M|_{N_y})^{-1}(\exp_M X|_{\Omega_{2,K_5}})^{-1}(y)
\end{aligned}$$

Hence the computation shows that we obtain a well-defined section of the tangent bundle on $\Omega_{\frac{5}{4},K_5}$ via

$$X^* : \Omega_{\frac{5}{4},K_5} \to TM, X^*(y) := (\exp_M|_{N_y})^{-1} \circ (\exp_M \circ X)^{-1}(y)$$

Let $(V_{5,n}, \kappa_n) \in \mathcal{F}_5(K_5)$ and $y \in V_{\frac{5}{4},n}$. Observe that $\exp_n|_{T\kappa_n(N_y)}$ is injective. Furthermore $F_{X^*_{[n]}}(\kappa_n(y)) = \exp_n(\kappa_n(y), X^*_{[n]}(\kappa_n(y)))$ and $(\{\kappa_n(y)\}, X^*_{[n]}(\kappa_n(y))) \in T\kappa_n(N_y)$ hold. This implies $(\exp_n|_{T\kappa_n(N_y)})^{-1} F_{X^*_{[n]}}(\kappa_n(y)) = (\kappa_n(y), X^*_{[n]}(\kappa_n(y)))$, whence the local identity above yields

$$X^*(y) := (\exp_M|_{N_y})^{-1} \circ F_X^{-1}(y) = T\kappa_n^{-1}(\mathrm{id}_{B_2(0)}, X^*_{[n]})\kappa_n(x) \text{ for each } y \in V_{\frac{5}{4},n}. \tag{D.0.15}$$

As $X^*_{[n]}$ is a smooth map by Lemma D.0.4, (D.0.15) shows that X^* is smooth. Hence $X^* \in \mathfrak{X}\left(\Omega_{\frac{5}{4},K_5}\right)$ follows. In addition for each $(V_{5,n}, \kappa_n) \in \mathcal{F}_5(K_5)$ by choice of ρ_n

$$\left\| X^*_{[n]} \right\|_{\overline{B_2(0)},1} < \rho_n = \min\{\nu, \tau\} < R. \tag{D.0.16}$$

holds. Define $\mathcal{H}_R := (\mathrm{res}^M_{\Omega_{5,K_5}})^{-1}(\mathcal{H}_R^{\Omega_{5,K_5}})$ and observe that the estimates obtained in Step 1 and 2 remain valid for sections in this set.

Conclusion: We have constructed C^1-neighborhoods of the zero-section

$$\mathcal{H}_R^{\Omega_{5,K_5}} := \Gamma^{-1}\left(\prod_{(V_{5,n},\kappa_n)\in\mathcal{F}_5(K_5)} \mathcal{H}_n\right) \subseteq \mathfrak{X}(\Omega_{5,K_5}),$$
$$\mathcal{H}_R := (\mathrm{res}^M_{\Omega_{5,K_5}})^{-1}(\mathcal{H}_R^{\Omega_{5,K_5}}) \subseteq \mathfrak{X}(M)$$

where $\Gamma\colon \mathfrak{X}(\Omega_{5,K_5}) \to \prod_{(V_{5,n},\kappa_n)\in\mathcal{F}_5(K_5)} C^\infty(V_{5,n}, \mathbb{R}^d)$ is the embedding defined in C.3.1 and each $\mathcal{H}_n \subseteq C^\infty(V_{5,n}, \mathbb{R}^d)$ is an open C^1-neighborhood of the zero map.
By construction $\mathcal{H}_R$ is contained in the zero-neighborhood $E_{5,K} \cap (\mathrm{res}^M_{\Omega_{5,K}})^{-1}(P)$ chosen in advance. Here $E_{5,K}$ is a neigbhorhood as in Lemma D.0.7 and $P \subseteq \mathfrak{X}(\Omega_{5,K})$ is an open C^1-neighborhood of the zero-section. In particular Lemma D.0.7 implies that each element of $\mathcal{H}_R$ satisfies the assertions of Lemma D.0.6 (d), i.e.:
For $(V_{5,n}, \kappa_n) \in \mathcal{F}(K_5)$ and $X \in \mathcal{H}_R$, we obtain $X_{\kappa_n}(\overline{V_{1,n}}) \subseteq B_\nu(0)$ with $B_2(0) \times B_\nu(0) \subseteq \mathrm{dom}\exp_n$.
For a pair $(X, Y) \in \mathcal{H}_R \times \mathcal{H}_R$ there are vector fields $X \diamond Y \in \mathfrak{X}(\Omega_{2,K_5})$ respectively $X^* \in \mathfrak{X}\left(\Omega_{\frac{5}{4},K_5}\right)$ such that the following identities are satisfied:

$$\exp_M \circ X \diamond Y = \exp_M X \exp_M Y|_{\Omega_{2,K_5}} \tag{D.0.17}$$
$$\exp_M \circ X^* = (\exp_M \circ X|_{\Omega_{2,K_5}})^{-1}|_{\Omega_{\frac{5}{4},K_5}} \tag{D.0.18}$$

We note that if X and Y are the zero section, then the local formulas (D.0.13) and (D.0.15) (with Lemma D.0.4 (e)) prove that $X \diamond Y$ respectively X^* are the zero section in $\mathfrak{X}(\Omega_{2,K_5})$ respectively $\mathfrak{X}\left(\Omega_{\frac{5}{4},K_5}\right)$.

The neighborhood constructed in this section is used in Section 6 to obtain symmetric neighborhoods in the space of compactly supported orbisections. The argument in Construction D.0.8

depends only on a finite atlas. Hence the sets constructed are open in $\mathfrak{X}(M)$ with the topology introduced in Definition C.3.1. Unfortunately the vector fields $X \diamond Y$ and X^* will thus in general **not** be defined on all of M. Because of this we are not able to prove a statement of the following kind: If $X, Y \in \mathcal{H}_R$ then $X \diamond Y \in E$ and $X^* \in E$. Instead we may at this moment only prove the following estimate:

D.0.9 Corollary *Consider the setting of Construction D.0.8 and let $\mathcal{H}_n$,$(V_{5,n}, \kappa_n) \in \mathcal{F}_5(K_5)$ and $\mathcal{H}_R$ be as constructed there. For each pair $\eta, \xi \in \mathcal{H}_n$, the map $\eta \diamond \xi \colon B_2(0) \to B_\tau(0)$ satisfies $\|\eta \diamond \xi\|_{\overline{B_1(0)},1} < \tau < R$. Hence by* (D.0.13) *for any pair $(X, Y) \in \mathcal{H}_R \times \mathcal{H}_R$ we derive $\left\|(X \diamond Y)_{[n]}\right\|_{\overline{B_1(0)},1} < \tau$ for each $(V_{5,n}, \kappa_n) \in \mathcal{F}_5(K_5)$.*

In Section 6 we consider a setting, which allows a unique extension of $X \diamond Y$ to all of M. In this case the Corollary will imply the result mentioned above (cf. Proposition 6.1.8).

Proof of Corollary D.0.9. By (D.0.14) it suffices to to prove that the norm of the derivative is bounded by τ. To do so we recall the estimates from Step 1 of Construction D.0.8: Let $x \in \overline{B_1(0)}, y \in B_2(0)$ and consider $\xi \in \mathcal{H}_R$. Then $F_\xi(x) \in B_2(0)$ and $\|\xi\|_{\overline{B_3(0)},1} < \mu_n$ hold with $0 < \mu_n < \min\left\{\nu, \frac{\sigma_n}{2}, \frac{\tau}{6dt_n}\right\}$. Recall that $\|d_1 f_n(y_1, y_2, y_3; \cdot)\|_{\mathrm{op}} < \frac{\tau}{3}$ holds and t_n is an upper bound for $\|df_n(y_1, y_2, y_3; \cdot)\|_{\mathrm{op}}$ with $(y_1, y_2, y_3) \in \overline{B_2(0)} \times \overline{B_{\mu_n}(0)} \times \overline{B_{\mu_n}(0)}$. As $\mathcal{H}_n \subseteq \mathcal{N}_n$ holds for an open neighborhood $\mathcal{N}_n$ constructed via Lemma D.0.4 we deduce from the proof of the Lemma $\frac{1}{4} \geq \|dF_\xi(x;\cdot) - \mathrm{id}_{\mathbb{R}^d}\|_{\mathrm{op}} \geq \|dF_\xi(x;\cdot)\|_{\mathrm{op}} - 1$ for $\|x\|_\infty < 3$. We obtain the estimate $\|d\xi(x;y)\|_\infty \leq \|d\xi(x;\cdot\|_{\mathrm{op}} < 2$ for each pair $(x, y) \in \overline{B_1(0)} \times \overline{B_1(0)}$ Using the rule on partial derivatives and the chain rule with these estimates we compute for $(x, y) \in \overline{B_1(0)} \times \overline{B_1(0)}$:

$$
\begin{aligned}
\|d(\eta \diamond \xi)(x;y)\|_\infty &\overset{(\mathrm{D.0.12})}{=} \|df_n(x, \xi(x), \eta(F_\xi(x)), y, d\xi(x,y), d\eta(F_\xi(x), dF_\xi(x,y)))\|_\infty \\
&\leq \|d_1 f_n(x, \xi(x), \eta(F_\xi(x)), y\|_\infty + \|df_n(x, \xi(x), \eta(F_\xi(x)); \cdot)\|_{\mathrm{op}} \cdot \|d\xi(x;y)\|_\infty \\
&\quad + \|df_n(x, \xi(x), \eta(F_\xi(x)); \cdot)\|_{\mathrm{op}} \cdot \|d\eta(F_\xi(x); \cdot)\|_{\mathrm{op}} \cdot \|dF_\xi(x;y)\|_\infty \\
&< \frac{\tau}{3} + \underbrace{\|df_n(x, \xi(x), \eta(F_\xi(x)); \cdot)\|_{\mathrm{op}}}_{\leq t_n} (\underbrace{\|d\xi(x;y)\|_\infty}_{\leq d\mu_n} + 2 \underbrace{\|d\eta(F_\xi(x); \cdot)\|_{\mathrm{op}}}_{\leq d\mu_n}) \\
&\leq \frac{\tau}{3} + \frac{\tau}{6} + \frac{\tau}{3} \leq \tau
\end{aligned}
$$

We derive $\left\|\frac{\partial}{\partial x_j} \eta \diamond \xi(x)\right\|_\infty < \tau$ for $x \in \overline{B_1(0)}$ and $i = 1, 2, \ldots, d$ and thus $\|\eta \diamond \xi\|_{\overline{B_1(0)},1} < \tau$. ◻

D.0.10 Lemma *Consider the open zero neighborhoods $\mathcal{H}_R$ as in Construction D.0.8. The maps*

$$
\begin{aligned}
&c \colon \mathcal{H}_R^{\Omega_{5,K}} \times \mathcal{H}_R^{\Omega_{5,K}} \to \mathfrak{X}(\Omega_{2,K_5}), \ (X, Y) \mapsto X \diamond Y \\
&\iota \colon \mathcal{H}_R^{\Omega_{5,K}} \to \mathfrak{X}\left(\Omega_{\frac{5}{4},K_5}\right), \ X \mapsto X^*
\end{aligned}
$$

are smooth.

Proof. Let I be the finite set indexing $\mathcal{F}_5(K_5)$. Following Definition C.3.1 and the definition of Ω_{r,K_5}, the topology on $\mathfrak{X}(\Omega_{r,K_5}), r \in [1,5]$ is defined via the linear embedding with closed image

$$\Gamma_r \colon \mathfrak{X}(\Omega_{r,K_5}) \to \prod_{k\in I} C^\infty(V_{r,k},\mathbb{R}^d) = \bigoplus_{k\in I} C^\infty(V_{r,k},\mathbb{R}^d).$$

Therefore the maps $p_k^r := \mathfrak{X}(\Omega_{r,K_5}) \to C^\infty(V_{r,k},\mathbb{R}^d), p_k^r(X) \mapsto X_{\kappa_k}|_{V_{r,k}}, k \in I$ define a patchwork for $\mathfrak{X}(\Omega_{r,K_5})$ indexed by I.
Define $p\colon \mathfrak{X}(\Omega_{5,K_5}) \times \mathfrak{X}(\Omega_{5,K_5}) \to \bigoplus_{k\in I} C^\infty(V_{5,k},\mathbb{R}^d)\times C^\infty(V_{5,k},\mathbb{R}^d), (X,Y) \mapsto (p_k^5 \times p_k^5(X,Y))_{k\in I}$.
Recall that finite products coincide with direct sums in the category of locally convex vector spaces. The universal property of the direct sum therefore assures that the map

$$L\colon \bigoplus_{k\in I} C^\infty(V_{5,k},\mathbb{R}^d)\times C^\infty(V_{5,k},\mathbb{R}^d) \to \left(\bigoplus_{k\in I} C^\infty(V_{5,k},\mathbb{R}^d)\right) \times \left(\bigoplus_{i\in I} C^\infty(V_{5,k},\mathbb{R}^d)\right)$$
$$(X_k,Y_k)_{k\in I} \mapsto ((X_k)_{k\in I},(Y_k)_{k\in I}))$$

is an isomorphism of locally convex spaces. Furthermore $L \circ p = \Gamma_5 \times \Gamma_5$ holds. As Γ_5 is an embedding with closed image, the map $\Gamma_5 \times \Gamma_5$ is a linear embedding with closed image (identifying the domain of Γ_5 via the embedding with a closed subspace of the codomain of Γ_5 this follows from [10, II, No. 6 Proposition 8]). We conclude that p is an embedding with closed image and the family $(p_k^5 \times p_k^5)_{k\in I}$ yields a patchwork for $\mathfrak{X}(\Omega_{5,K_5}) \times \mathfrak{X}(\Omega_{5,K_5})$.
We claim that the maps c and ι are patched mappings which are smooth on the patches. If this were true, then the assertion follows from Proposition C.3.7. Proceed in two steps and prove the claim first for the map c:
Recall from Construction D.0.8 that $\mathcal{H}_R^{\Omega_{5,K_5}} = \bigcap_{n\in I}(\theta_{\kappa_n}^{\Omega_{5,K_5}})^{-1}(\mathcal{H}_n)$ holds. Here each of the sets $\mathcal{H}_n$ is an open neighborhood of the zero-map with $\mathcal{H}_n \subseteq C^\infty(\kappa_n^{-1},\mathbb{R}^d)^{-1}(\mathcal{H}'_n) = C^\infty(\kappa_n^{-1},\mathbb{R}^d)(\mathcal{H}'_n)$ and $\mathcal{H}'_n \subseteq C^\infty(B_5(0),\mathbb{R}^d)$. We define maps

$$h_n \colon \mathcal{H}'_n \times \mathcal{H}'_n \to C^\infty(B_2(0),\mathbb{R}^d), (\eta,\xi) \mapsto \eta \diamond \xi$$
$$c_n \colon \mathcal{H}_n \times \mathcal{H}_n \to C^\infty(V_{2,n},\mathbb{R}^d), (X,Y) \mapsto C^\infty(\kappa_n,\mathbb{R}^d) \circ h_n \circ (C^\infty(\kappa_n^{-1},\mathbb{R}^d) \times C^\infty(\kappa_n^{-1},\mathbb{R}^d))(X,Y).$$

Observe that by Step 1 in Construction D.0.8 each map c_n maps the zero map $(0,0) \in \mathcal{H}_n \times \mathcal{H}_n$ to $0 \in C^\infty(V_{2,n},\mathbb{R}^d)$. From the definition of c and the identity (D.0.13) a trivial computation yields the identity $c_n \circ p_n^5 = p_n^2 \circ c$ for each $n \in I$. Therfore c is a patched mapping, whose compatible family is $(c_n)_{n\in I}$. By Proposition C.3.7 the first part of the claim will hold if each c_n is a smooth map. However c_n will be smooth if and only if $h_n \colon \mathcal{H}'_n \times \mathcal{H}'_n \to C^\infty(B_2(0),\mathbb{R}^d), (\eta,\xi) \mapsto \eta \diamond \xi$ is smooth, since $C^\infty(\kappa_n^{-1},\mathbb{R}^d)$ and $C^\infty(\kappa_n,\mathbb{R}^d)$ are mutually inverse isomorphisms of topological vector spaces by [25, Lemma A.1]. Fix $n \in I$ and prove that h_n is a smooth map:
To this end recall the constants ε_n, δ_n obtained in Construction D.0.8. By Lemma D.0.3 we may consider the smooth maps

$$e_n \colon B_4(0) \times B_{\varepsilon_n}(0) \to \mathbb{R}^d, (x,y) \mapsto \exp_n(x,y)$$
$$a_n \colon B_4(0) \times B_{\delta_n}(0) \to B_{\varepsilon_n}(0), a(x,y) := b_n(x,x+y).$$

By [25, Proposition 4.23 (a)] these maps induce smooth push-forward maps

$$e_{n*}\colon \lfloor \overline{B_3(0)}, B_{\varepsilon_n}(0)\rfloor \to C^\infty(B_3(0),\mathbb{R}^d), e_{n*}(\gamma)(x) := e_n(x,\gamma(x))$$
$$a_{n*}\colon \lfloor \overline{B_2(0)}, B_{\delta_n}(0)\rfloor_\infty \to C^\infty(B_2(0),\mathbb{R}^d), a_{n*}(\eta)(x) := a_n(x,\eta(x)),$$

where $\lfloor \overline{B_3(0)}, B_{\varepsilon_n}(0)\rfloor_\infty \subseteq C^\infty(B_5(0),\mathbb{R}^n)$ and $\lfloor \overline{B_2(0)}, B_{\delta_n}(0)\rfloor_\infty \subseteq C^\infty(B_{\frac{21}{10}}(0),\mathbb{R}^d)$ are open sets. Recall from Construction D.0.8 that $\mathcal{H}'_n$ is a subset of an open set $\mathcal{N}_n$ which has been constructed by an application of Lemma D.0.4. Hence $\eta \in H'_n$ satisfies the estimate (D.0.3), whence $\eta(\overline{B_3(0)}) \subseteq B_{\varepsilon_n}(0)$ holds. In other words $\mathcal{H}'_n \subseteq \lfloor \overline{B_3(0)}, B_{\varepsilon_n}(0)\rfloor_\infty$ is satisfied. By definition the identity $e_{n*}(\eta) = F_\eta$ holds with F_η as defined in Lemma D.0.4. Furthermore applying the estimate (D.0.3) again we obtain $e_{n*}(\eta) \in \lfloor \overline{B_2(0)}, B_3(0)\rfloor_\infty$. By [25, Lemma 11.4] there is a smooth composition map

$$\Theta\colon C^\infty(B_3(0),\mathbb{R}^d) \times \lfloor \overline{B_2(0)}, B_3(0)\rfloor_\infty \to C^\infty(B_2(0),\mathbb{R}^d), (f,g) \mapsto f \circ g|_{B_2(0)}.$$

We conclude that the composition $\Theta \circ (e_{n*} \times e_{n*})\colon \mathcal{H}'_n \times \mathcal{H}'_n \to C^\infty(B_2(0),\mathbb{R}^d)$ is well defined and smooth. By definition of $\mathcal{H}'_n$, we derive for $\eta \in H'_n$ the estimate $F_\eta(x) \in B_{\frac{\delta_n}{2}}(x)$ for $x \in B_3(0)$ (see Lemma D.0.4 (a)). Thus $\Theta(e_{n*}(\eta), e_{n*}(\xi))(x) - x \in B_{\delta_n}(0)$ holds for $x \in \overline{B_2(0)}, \eta, \xi \in \mathcal{H}'_n$.
Combine the identity (D.0.12) with the definition of f_n in Lemma D.0.3 (c) to deduce the identity

$$h_n(\eta,\xi) = a_{n*}(\Theta(e_{n*}(\eta), e_{n*}(\xi)) - \mathrm{id}_{B_2(0)}).$$

We conclude that h_n is smooth as composition of smooth maps. Summing up, this proves the first part of the claim.
As a second step we construct a compatible family for ι. To this end define maps

$$i_n\colon \mathcal{H}'_n \to C^\infty(B_{\frac{5}{4}}(0),\mathbb{R}^d), \xi \mapsto \xi^*|_{B_{\frac{5}{4}}(0)}$$
$$\iota_n\colon \mathcal{H}_n \to C^\infty(V_{2,n},\mathbb{R}^d), X \mapsto C^\infty(\kappa_n,\mathbb{R}^d) \circ i_n \circ C^\infty(\kappa_n^{-1},\mathbb{R}^d).$$

From the identity (D.0.15) we derive $p_n^{\frac{5}{4}}\iota = \iota_n p_n^5$. Hence ι is a patched mapping and we have to prove that each ι_n is smooth. Again ι_n will be smooth if i_n is smooth.
Recall that $\mathcal{H}'_n \subseteq \mathcal{N}_n$ holds for an open set $\mathcal{N}_n \subseteq C^\infty(B_5(0),\mathbb{R}^d)$ with the properties of the set $\mathcal{N}$ in Lemma D.0.4. Hence the map $I_n\colon \mathcal{N}_n \to C^\infty(B_2(0),\mathbb{R}^d), \xi \mapsto \xi^*|_{B_2(0)}$ is smooth by Lemma D.0.4 (f). Let $\lambda\colon B_{\frac{5}{4}}(0) \hookrightarrow B_2(0)$ be the canonical inclusion. The pullback $C^\infty(\lambda,\mathbb{R}^d)$ is continuous linear, whence smooth. Finally the identity $i_n = C^\infty(\lambda,\mathbb{R}^d) \circ I_n|_{H'_n}$ assures that i_n is smooth. □

E. Maps of orbifolds

In this section we recall the notion of an orbifold map in local charts which was introduced in [51]. (cf. Section 2.3 for details on Orbifold). Our exposition follows [51] and we repeat basic facts for the readers convenience. Orbifold maps in the sense discussed here correspond to maps in a category of groupoids. The notion of orbifold map developed here is thus equivalent to other types of orbifold maps which are equivalent to maps in the associated groupoid category (cf. [13] for the so called Chen-Ruan good map respectively [1] for the Moerdijk-Pronk strong map).

E.1. (Quasi-)Pseudogroups

In this section we let M be a smooth manifold.

E.1.1 Notation (Transitions) A *transition on M* is a diffeomorphism $f\colon U \to V$ where U, V are open subsets of M. Notice that the empty map $\emptyset \to \emptyset$ is a transition on M.
The product of two transitions $f\colon U \to V$, $g\colon U' \to V'$ is the transition

$$f|^{f(U\cap V')} \circ g|_{g^{-1}(U\cap V')}\colon g^{-1}(U\cap V') \to f(U\cap V'), x \mapsto f(g(x))$$

The inverse of f is the inverse of f as a function. If $f\colon U \to V$ is a map, we denote by $\operatorname{dom} f$ the *domain* of f and $\operatorname{cod} f$ the *codomain* of f. For $x \in \operatorname{dom} f$ denote by $\operatorname{germ}_x f$ the *germ of f at x* and by $\mathcal{A}(M)$ we denote the set of all transitions of M.

E.1.2 Definition (Pseudogroup) A *pseudogroup* on M is a subset $P \subseteq \mathcal{A}(M)$ which is closed under products and inversion of transitions. We call P a *full pseudogroup*, if for every open subset $U \subseteq M$ the transition id_U is contained in P. A full pseudgroup is called *complete* if it satisfies

(*Gluing Property*) For each $f \in \mathcal{A}(M)$ and open covering $(U_i)_{i\in I}$ of $\operatorname{dom} f$ with $f|_{U_i} \in P$ for all $i \in I$, then f is an element of P.

The pseudogroup P is *closed under restrictions*, if for any $f \in P$ and open set $U \subseteq \operatorname{dom} f$, the map $f|_U^{f(U)}\colon U \to f(U)$ is in P.

E.1.3 Definition (Quasi-Pseudogroup) A subset P of $\mathcal{A}(M)$ is called a *quasi-pseudogroup* on M if the following properties are satisfied:

(a) For each $f \in P$ and $x \in \operatorname{dom} f$, there exist an open set U with $x \in U \subseteq \operatorname{dom} f$ and $g \in P$ together with an open set V such that $f(x) \in V \subseteq \operatorname{dom} g$ and

$$(f|_U)^{-1} = g|_V.$$

(b) If $f, g \in P$ and $x \in f^{-1}(\operatorname{cod} f \cap \operatorname{dom} g)$, then there exists $h \in P$ and an open neighborhood $U \subseteq f^{-1}(\operatorname{cod} f \cap \operatorname{dom} g) \cap \operatorname{dom} h$ of x with $g \circ f|_U = h|_U$.

Identities like inversion and composition of elements in a quasi-pseudogroup are only required to correspond locally to other elements in the quasi-pseudogroup. For pseudogroups these identities have to globally correspond to elements of the pseudogroup. Quasi-pseudogroups are designed to work with the germs of their elements. In general quasi-pseudogroups may be thought of as generators for pseudogroups in the following sense:

E.1.4 Definition Let P be a pseudogroup on M, which satisfies the gluing property and is closed under restrictions. The pseudogroup P is *generated* by a set $A \subseteq \mathcal{A}(M)$ if $A \subseteq P$ holds and for each $f \in P$ and $x \in \operatorname{dom} P$ there exists $g \in A$ and an open set $U \subseteq \operatorname{dom} f \cap \operatorname{dom} g$ with $x \in U$ and $f|_U = g|_U$.
Consider a subset B of $\mathcal{A}(M)$. If there exists a unique pseudogroup Q on M which satisfies the gluing property, is closed under restrictions and generated by B, we say B *generates* Q.

E.1.5 Remark (a) The set $\mathcal{A}(M)$ is a pseudogroup. Each pseudogroup is a quasi-pseudogroup.

(b) Each quasi-pseudogroup generates a unique pseudogroup, which satisfies the gluing property and is closed under restrictions. Vice versa each generating set for such a pseudogroup is necessarily a quasi-pseudogroup.

E.2. Charted orbifold maps

In this Section we let $(Q,\mathcal{U})$ and $(Q',\mathcal{U}')$ be orbifolds. Morphisms of orbifolds will be constructed in several steps, since they arise as equivalence classes of certain objects:

E.2.1 Definition Let $\mathcal{V} := \{ (V_i, G_i, \pi_i) | i \in I \}$ be a representative of $\mathcal{U}$. We abbreviate the disjoint union of the the chart domains of elements in $\mathcal{V}$ with

$$V := \coprod_{i \in I} V_i \quad \text{and set} \quad \pi \colon V \to Q, x \mapsto \pi_i(x) \ \forall x \in V_i$$

Then the subset

$$\mathbf{\Psi}(\mathcal{V}) := \{ f \in \mathcal{A}(V) | \pi \circ f = \pi|_{\operatorname{dom} f} \}$$

of all transitions on V is a complete pseudogroup on V, which is closed under restrictions.

The last definition may be used to associate to each orbifold an ètale Lie groupoid (as is explained in [51, 2.9 and 2.10]). Since we are not interested in the correspondence of orbifolds and Lie groupoids, we will not pursue this relation any further. However this relation were invaluable to derive the notion of orbifold map introduced in this section. We refer to [51] for further details.

E.2.2 Definition Let $f \colon Q \to Q'$ be a continuous map. Consider two orbifold charts $(V, G, \pi) \in \mathcal{U}$ and $(V', G', \pi') \in \mathcal{U}'$. A smooth map $f_V \colon V \to V'$ is called *local lift of f with respect to* (V, G, π) *and* (V', G', π') if $\pi' \circ f_V = f \circ \pi_V$ holds. In this case f_V is also called a local lift of f at q for each $q \in \pi(V)$.

E.2.3 Definition (Representative of an orbifold map) A *representative of an orbifold map* from an orbifold $(Q,\mathcal{U})$ to an orbifold $(Q',\mathcal{U}')$ is a tuple

$$\hat{f} := (f, \{ f_i \}_{i\in I}, P, \nu)$$

where

(R1) $f\colon Q \to Q'$ is a continuous map,
(R2) for each $i \in I$, the map $f_i\colon V_i \to V_i'$ is a local lift of f with respect to orbifold charts $(V_i, G_i, \pi_i) \in \mathcal{U}$, $(V_i', G_i', \pi_i') \in \mathcal{U}'$ such that

$$\bigcup_{i\in I} \pi_i(V_i) = Q$$

and $(V_i, G_i, \pi_i) \neq (V_j, G_j, \pi_j)$ holds for $i, j \in I,\ i \neq j$,
(R3) P is a quasi-pseudogroup which consists of changes of charts of the orbifold atlas

$$\mathcal{V} := \{ (V_i, G_i, \pi_i) | i \in I \}$$

of $(Q,\mathcal{U})$ and generates $\mathbf{\Psi}(\mathcal{V})$,
(R4) Set $F := \coprod_{i\in I} f_i\colon V = \coprod_{i\in I} V_i \to \coprod_{i\in I} V_i', x \mapsto f_i(x)$ if $x \in V_i$. Then $\nu\colon P \to \mathbf{\Psi}(\mathcal{U}')$ is a map which assigns to each $\lambda \in P$ a change of charts morphism

$$\nu(\lambda)\colon (W', H', \chi') \to (V', G', \varphi')$$

between orbifold charts in $\mathcal{U}'$ such that the following properties are satisfied
a) $F \circ \lambda = \nu(\lambda) \circ F|_{\operatorname{dom}\lambda}$,
b) for all $\lambda, \mu \in P$ and all $x \in \operatorname{dom}\lambda \cap \operatorname{dom}\mu$ with $\operatorname{germ}_x \lambda = \operatorname{germ}_x \mu$ we have

$$\operatorname{germ}_{F(x)} \nu(\lambda) = \operatorname{germ}_{F(x)} \nu(\mu),$$

c) for all $\lambda, \mu \in P$, $x \in \lambda^{-1}(\operatorname{cod}\lambda \cap \operatorname{dom}\mu)$ we have

$$\operatorname{germ}_{F(\lambda(x))} \nu(\mu) \cdot \operatorname{germ}_{F(x)} \nu(\lambda) = \operatorname{germ}_{F(x)} \nu(h)$$

where h is an element of P such that there is an open set U with

$$x \in U \subseteq \lambda^{-1}(\operatorname{cod}\lambda \cap \operatorname{dom}\mu) \cap \operatorname{dom} h$$

and $\mu \circ \lambda|_U = h|_U$,
d) for all $\lambda \in P$ and $x \in \operatorname{dom}\lambda$ such that there is an open set $x \in U \subseteq \operatorname{dom}\lambda$ with $\lambda|_U = \operatorname{id}_U$ we have $\operatorname{germ}_{F(x)} \nu(\lambda) = \operatorname{germ}_{F(x)} \operatorname{id}_{U'}$ where $U' := \coprod_{i\in I} V_i'$.

The orbifold atlas $\mathcal{V}$ is called the *domain atlas* of the representative $\hat{f}$, and the set $\{ (V_i', G_i', \pi_i') | i \in I \}$ is called the *range family* of $\hat{f}$. Note that the range family is not necessarily indexed by I. The continuous map f will sometimes be called the *underlying map* of the representative $\hat{f}$. The map f may not be chosen arbitrarily. As [51, Example 4.5] shows, it is not even sufficient to require that f be a homeomorphism, to assure that there is a representative $\hat{f}$ with underlying map f.

The technical condition in (R2) that two orbifold charts in $\mathcal{V}$ be distinct is required, because in several places I is used as an index set for $\mathcal{V}$ (cf. property (R3)).
In view of the Definition E.2.3, it is useful to have a shorthand for the change of charts associated to a given orbifold atlas. Hence we fix the following way of speaking.

E.2.4 Notation Let $\mathcal{V} = \{ (V_i, G_i, \psi_i) | i \in I \}$ be a representative of $\mathcal{U}$. Recall from Lemma 4.1.4 the notation

$$\mathcal{C}h_{V_i,V_j} := \{ \lambda \colon V_i \supseteq \operatorname{dom} \lambda \to \operatorname{cod} \lambda \subseteq V_j | \lambda \text{ is a change of charts} \}$$

We define the *set of all change of charts* of the atlas $\mathcal{V}$ via

$$\mathcal{C}h_{\mathcal{V}} := \{ \lambda \colon V_i \supseteq \operatorname{dom} \lambda \to \operatorname{cod} \lambda \subseteq V_j | \lambda \text{ is a change of charts }, i, j \in I \} = \bigcup_{(i,j) \in I \times I} \mathcal{C}h_{V_i,V_j} .$$

Observe that $\mathcal{C}h_{\mathcal{V}}$ is a (quasi-)pseudogroup by Proposition 2.2.2, which generates $\mathbf{\Psi}(\mathcal{V})$.

E.2.5 Definition Let $\hat{f} := (f, \{ f_i \}_{i \in I}, P_1, \nu_1)$ and $\hat{g} := (g, \{ g_i \}_{i \in I}, P_2, \nu_2)$ be two representatives of orbifold maps with the same domain atlas $\mathcal{V}$ representing the orbifold structure $\mathcal{U}$ on Q and both range families being contained in the orbifold atlas $\mathcal{V}'$ of $(Q', \mathcal{U}')$. Set $F := \coprod_{i \in I} f_i$. We say that $\hat{f}$ is *equivalent* to $\hat{g}$ if $f = g$, $f_i = g_i$ for all $i \in I$ and

$$\operatorname{germ}_{F(x)} \nu_1(\lambda_1) = \operatorname{germ}_{F(x)} \nu_2(\lambda_2)$$

holds for all $\lambda_1 \in P_1, \lambda_2 \in P_2, x \in \operatorname{dom} \lambda_1 \cap \operatorname{dom} \lambda_2$ with $\operatorname{germ}_x \lambda_1 = \operatorname{germ}_x \lambda_2$. This defines an equivalence relation the equivalence class of $\hat{f}$ will be denoted by

$$(f, \{ f_i \}_{i \in I}, [P_1, \nu_1]).$$

By abuse of notation we denote by $\hat{f}$ the equivalence class $[\hat{f}]$ of the representative $\hat{f}$, if it is clear that we refer to equivalence classes. The equivalence class of the representative $\hat{f}$ is called *orbifold map with domain atlas $\mathcal{V}$ and range atlas $\mathcal{V}'$*, in short *orbifold map with* $(\mathcal{V}, \mathcal{V}')$ or, if the specific atlases are not important, a *charted orbifold map*. Define $\operatorname{Orb}(\mathcal{V}, \mathcal{V}')$ to be *the set of all orbifold maps with* $(\mathcal{V}, \mathcal{V}')$. To shorten our notation we denote an element $\hat{h} \in \operatorname{Orb}(\mathcal{V}, \mathcal{V}')$ by $\mathcal{V} \xrightarrow{\hat{h}} \mathcal{V}'$

E.2.6 Remark

(a) In [51] all orbifolds are assumed to be second countable, since second countability is the default setting for Lie-groupoids (cf. [48]). We only required orbifolds to be paracompact. Reviewing the arguments in [51], it is easy to see that all constructions there remain valid under the weaker assumption of paracompactness. Furthermroe [12,32] and the survey article by Lerman [44] outline the theory of Lie-groupoids for non second countable manifolds. In particular the article by Lerman indicates that all desirable properties on the groupoid side are preserved for paracompact orbifolds and manifolds. Hence we require only the weaker condition.

(b) In Definition E.2.3 we used quasi-pseudogroups instead of the pseudogroups $\mathcal{C}h_{\mathcal{V}}$ or $\mathbf{\Psi}(\mathcal{V})$ since in general, a quasi-pseudogroup P will be much smaller sometimes even finite. Observe the following facts, whose proofs we ommit here:

   i. Let $(f, \{ f_i \}_{i \in I}, P, \nu)$ be a representative of an orbifold map. Replacing P with a quasi-psudogroup P' whose elements arise as restrictions of maps in P (if necessary reducing them to open neighborhoods which are stable with respect to the group action), one may replace ν with a map ν', which maps each elements in P' to an open embedding in the range atlas. The pair (P', ν') may be chosen such that $(f, \{ f_i \}_{i \in I}, P, \nu)$ and $(f, \{ f_i \}_{i \in I}, P', \nu')$ are in the same equivalence class.

   ii. Consider a representative of an orbifold map $\hat{f} \colon (Q,\mathcal{U}) \to \mathcal{M}$, where $\mathcal{M}$ is a connected manifold (without boundary). The map ν may then always be chosen as the map taking $h \in P$ to $\mathrm{id}_{\mathcal{M}}$.

E.3. The identity morphism

In this section we construct the identity morphism in the category of reduced orbifolds.

E.3.1 Definition Let $f \colon Q \to Q'$ be a continuous map between orbifolds $(Q,\mathcal{U}), (Q',\mathcal{U}')$. Suppose f_V is a local lift with respect to the orbifold charts $(V, G, \pi) \in \mathcal{U}$ and $(V', G', \pi') \in \mathcal{U}'$. Consider embeddings of orbifold charts in $\mathcal{U}$ respectively $\mathcal{U}'$

$$\lambda \colon (W, K, \chi) \to (V, G, \pi) \quad \text{and} \quad \mu \colon (W', K', \chi') \to (V', G', \pi'),$$

such that $f_V(\lambda(W)) \subseteq \mu(W')$ holds. Then the map

$$g := \mu^{-1} \circ f_V \circ \lambda \colon W \to W'$$

is a local lift of f with respect to (W, K, χ) and (W', K', χ'). We say f_V induces the local lift g with respect to λ and μ and call g *induced lift of f with respect to f_V, λ and μ.*

E.3.2 Proposition ([51, Proposition 5.3]) *Let $(Q,\mathcal{U})$ be an orbifold and f_V be a local lift of* id_Q *with respect to $(V, G, \pi), (V', G', \pi') \in \mathcal{U}$. For each $v \in V$ there exists a restriction $(S, G_S, \pi|_S)$ of (V, G, π) with $v \in V$ and a restriction $(S', G', \pi|'_{S'})$ of (V', G', π') such that $f_V|_S^{S'}$ is diffeomorphism which is a change of charts from $(S, G_S, \pi|_S)$ to $(S', G', \pi'|_{S'})$. In particular, $f_V|_S$ induces the identity* id_S *with respect to the embeddings of orbifold charts* id_S *and* $(f_V|_S)^{-1}$.

Proposition E.3.2 shows that every local lift of the identity id_Q is a local diffeomorphism (but in general it need not be a global diffeomorphism as [51, Example 5.4] shows).

E.3.3 Proposition ([51, Proposition 5.5]) *Let $(Q,\mathcal{U})$ be an orbifold and $\{ f_i \}_{i \in I}$ a family of local lifts of* id_Q *which satisfies (R2). Then there exists a pair (P, ν), such that $(\mathrm{id}_Q, \{ f_i \}_{i \in I}, (P, \nu))$ is a representative of an orbifold map on $(Q,\mathcal{U})$. The pair (P, ν) is unique up to equivalence of representatives of orbifold maps.*

E.3.4 Proposition ([51, Proposition 5.6]) *Let Q be a topological space and suppose $\mathcal{U}$ and $\mathcal{U}'$ are orbifold structures on Q. Consider a charted orbifold map*

$$\hat{f} = (\mathrm{id}_Q, \{ f_i \}_{i\in I}, [P, \nu])$$

such that the domain atlas $\mathcal{V}$ is a representative of $\mathcal{U}$ and the range family $\mathcal{V}'$, which is an orbifold atlas, is a representative of $\mathcal{U}'$. Furthermore for each $i \in I$, the map f_i is a local diffeomorphism. Then $\mathcal{U} = \mathcal{U}'$ holds, i.e. the orbifolds coincide.

E.3.5 Definition Let $(Q,\mathcal{U})$ be an orbifold and $\hat{f} = (f, \{ f_i \}_{i\in I}, [P, \nu])$ be a charted orbifold map whose domain atlas is a representative of $\mathcal{U}$. The representative $\hat{f}$ is called *lift of the identity* $\mathrm{id}_{(Q,\mathcal{U})}$ if $f = \mathrm{id}_Q$ holds and f_i is a local diffeomorphism for each $i \in I$. We also say that $\hat{f}$ is a representative of $\mathrm{id}_{(Q,\mathcal{U})}$. The set of all lifts of $\mathrm{id}_{(Q,\mathcal{U})}$ is the *identity morphism* $\mathrm{id}_{(Q,\mathcal{U})}$ of $(Q,\mathcal{U})$.

E.4. Composition of charted orbifold maps

E.4.1 Construction Let $(Q,\mathcal{U})$, $(Q',\mathcal{U}')$ and $(Q'',\mathcal{U}'')$ be orbifolds, and

$$\mathcal{V} := \{ (V_i, G_i, \pi_i) | i \in I \}, \quad \mathcal{V}' := \left\{ (V'_j, G'_j, \pi'_j) \middle| j \in J \right\}$$

be representatives of $\mathcal{U}$ respectively of $\mathcal{U}'$. Furthermore let $\mathcal{V}'' \subseteq \mathcal{U}''$, and $\mathcal{V}$ be indexed by I respectively $\mathcal{V}'$ be indexed by J. Consider charted orbifold maps

$$\hat{f} = (f, \{ f_i \}_{i\in I}, [P_f, \nu_f]) \in \mathrm{Orb}(\mathcal{V}, \mathcal{V}')$$

and

$$\hat{g} := (g, \{ g_j | j \in J \}, [P_g, \nu_g]) \in \mathrm{Orb}(\mathcal{V}', \mathcal{V}'').$$

Define $\alpha \colon I \to J$ to be the unique map such that for each $i \in I$, f_i is a local lift of f with respect to (V_i, G_i, π_i) and $(V'_{\alpha(i)}, G'_{\alpha(i)}, \pi'_{\alpha(i)})$. We define the composition of $\hat{g}$ and $\hat{f}$:

$$\hat{g} \circ \hat{f} := \hat{h} = (h, \{ h_i | i \in I \}, [P_h, \nu_h]) \in \mathrm{Orb}(\mathcal{V}, \mathcal{V}'')$$

is given by $h := g \circ f$ and $h_i := g_{\alpha(i)} \circ f_i$ for all $i \in I$. To construct a representative (P_h, ν_h) of $[P_h, \nu_h]$ fix representatives (P_f, ν_f) and (P_g, ν_g) of $[P_f, \nu_f]$ respectively $[P_g, \nu_g]$. Consider $\mu \in P_f$ with $\mathrm{dom}\,\mu \subseteq V_i$, $\mathrm{cod}\,\mu \subseteq V_j$ for the orbifold charts (V_i, G_i, π_i) and (V_j, G_j, π_j) in $\mathcal{V}$. Property (R4a) assures

$$f_j \circ \mu = \nu_f(\mu) f_i|_{\mathrm{dom}\,\mu},$$

where $\nu_f(\mu) \in \mathbf{\Psi}(\mathcal{U})$. Shrinking domains, we may assume without loss of generality that $\nu_f(\mu)$ is an element of $\mathbf{\Psi}(\mathcal{V}')$. For $x \in \mathrm{dom}\,\mu$ set $y_x := f_i(x) \in \mathrm{dom}\,\nu_f(\mu)$. Since P_g generates $\mathbf{\Psi}(\mathcal{V}')$ we may choose $\xi_{\mu,x} \in P_g$ such that there is an open set $y_x \in U'_{\mu,x} \subseteq \mathrm{dom}\,\xi_{\mu,x} \cap \mathrm{dom}\,\nu_f(\mu)$ and the following is satisfied:

$$\xi|_{U'_{\mu,x}} = \nu_f(\mu)|_{U'_{\mu,x}}$$

We may choose an open set $x \in U_{\mu,x} \subseteq \operatorname{dom}\mu$ such that $f_i(U_{\mu,x}) \subseteq U'_{\mu,x}$ holds. By adjusting choices one may achieve that for $\mu_1, \mu_2 \in P_f$ and $x_k \in \operatorname{dom}\mu_k,\ k = 1, 2$ we either have

$$\mu_1|_{U_{\mu_1,x_1}} \neq \mu_2|_{U_{\mu_2,x_2}} \quad \text{or} \quad \xi_{\mu_1,x_1} = \xi_{\mu_2,x_2} \tag{E.4.1}$$

Define the quasi-pseudogroup

$$P_h := \left\{ \mu|_{U_{\mu,x}} \middle| \mu \in P_f,\ x \in \operatorname{dom}\mu \right\}$$

and observe that it generates $\mathbf{\Psi}(\mathcal{V})$ as P_f generates $\mathbf{\Psi}(\mathcal{V})$. As property (E.4.1) holds, we obtain a well defined map

$$\nu_h \colon P_h \to \mathbf{\Psi}(\mathcal{U}''), \nu_h(\mu|_{U_{\mu,x}}) := \nu_g(\xi_{\mu,x}).$$

Since ν_g and ν_f satisfy the properties (R4a) - (R4d) the same holds for ν_h. Furthermore, the equivalence class of (P_h, ν_h) does not depend on the choices in the construction of P_h and ν_h.

So far we have only explained the composition of charted orbifold maps in $\operatorname{Orb}(\mathcal{V}, \mathcal{V}')$ and $\operatorname{Orb}(\mathcal{V}', \mathcal{V}'')$. Obviously we need the composition of maps in $\operatorname{Orb}(\mathcal{V}, \mathcal{V}')$ and maps in $\operatorname{Orb}(\mathcal{V}'', \mathcal{V}''')$ for arbitrary $\mathcal{V}', \mathcal{V}''$. The leading idea is to construct a common refinement of the range family and the atlas $\mathcal{V}''$ together with induced maps, which may then be composed as in Construction E.4.1. Before we introduce the general construction, we define the notion of induced charted orbifold maps:

E.4.2 Lemma and Definition ([51, Lemma and Defintion 5.11]) *Let $(Q,\mathcal{U})$ and $(Q',\mathcal{U}')$ be orbifolds. Consider representatives*

$$\begin{aligned} \mathcal{V} &= \{ (V_i, G_i, \pi_i) | i \in I \} \ \textit{of } \mathcal{U} \textit{ indexed by } I \\ \mathcal{V}' &= \{ (V'_l, G'_l, \pi'_l) | l \in L \} \ \textit{of } \mathcal{U}' \textit{ indexed by } L, \textit{ and a charted map} \\ \hat{f} &= (f, \{ f_i \}_{i \in I}, [P_f, \nu_f]) \in \operatorname{Orb}(\mathcal{V}, \mathcal{V}'). \end{aligned}$$

Define $\beta \colon I \to L$ to be the unique map such that for each $i \in I$, f_i is a local lift of f with respect to (V_i, G_i, π_i) and $(V'_{\beta(i)}, G'_{\beta(i)}, \pi'_{\beta(i)})$. Suppose there are

- *a representative $\mathcal{W} = \{ W_j, H_j, \psi_j) | j \in J \}$ of $\mathcal{U}$, indexed by J,*
- *a subset $\left\{ (W'_j, H'_j, \psi'_j) \middle| j \in J \right\}$ of $\mathcal{U}'$, indexed by J (not necessarily an orbifold atlas),*
- *a map $\alpha \colon J \to I$,*
- *for each $j \in J$, an open embedding of orbifold charts*

$$\lambda_j \colon (W_j, H_j, \psi_j) \to (V_{\alpha(j)}, G_{\alpha(j)}, \pi_{\alpha(j)})$$

and an open embedding of orbifold charts

$$\mu_j \colon (W'_j, H'_j, \psi'_j) \to (V'_{\beta(\alpha(j))}, G'_{\beta(\alpha(j))}, \pi'_{\beta(\alpha(j))})$$

such that $f_{\alpha(j)}(\lambda_j(W_j) \subseteq \mu_j(W'_j)$ holds.

For each $j \in J$ we define the smooth map

$$h_j := \mu_j^{-1} \circ f_{\alpha(j)} \circ \lambda_j \colon W_j \to W'_j$$

Then the following assertions hold

(a) $\varepsilon := (\mathrm{id}_Q, \{\, \lambda_j \,\}_{j \in J}, [P_\varepsilon, \nu_\varepsilon])$ *(with $[P_\varepsilon, \nu_\varepsilon]$ provided by Proposition E.3.3) is a lift of* $\mathrm{id}_{(Q,\mathcal{U})}$.

(b) *The set $\left\{ (W'_j, H'_j, \psi'_j) \middle| j \in J \right\}$ and the family $\{\, \mu_j \,\}_{j \in J}$ may be extended to a representative*

$$\mathcal{W}' = \{\, (W'_k, H'_k, \psi'_k) | k \in K \,\}$$

of $\mathcal{U}'$ and a family of open embeddings $\{\, \mu_k \,\}_{k \in K}$ such that

$$\varepsilon' := (\mathrm{id}_{Q'}, \{\, \mu_k \,\}_{k \in K}, [P_{\varepsilon'}, \nu_{\varepsilon'}] \in \mathrm{Orb}(\mathcal{W}', \mathcal{V}')$$

(with $[P_{\varepsilon'}, \nu_{\varepsilon'}]$ provided by Proposition E.3.3) is a lift of the identity $\mathrm{id}_{(Q',\mathcal{U}')}$.

(c) *There is a uniquely determined equivalence class $[P_h, \nu_h]$ such that*

$$\hat{h} := (f, \{\, h_j \,\}_{j \in J}, [P_h, \nu_h]) \in \mathrm{Orb}(\mathcal{W}, \mathcal{W}')$$

and $\hat{f} \circ \varepsilon = \varepsilon' \circ \hat{h}$ holds.

We say that the charted orbifold map $\hat{h}$ is induced by $\hat{f}$.

E.4.3 Definition Let $(Q,\mathcal{U})$ and $(Q',\mathcal{U}')$ be orbifolds. Further let $\mathcal{V}_1, \mathcal{V}_2$ be representatives of $\mathcal{U}$ and $\mathcal{V}'_1, \mathcal{V}'_2$ be representatives of $\mathcal{U}'$. Suppose that $\hat{f}_i \in \mathrm{Orb}(\mathcal{V}_i, \mathcal{V}'_i),\ i = 1, 2$. We call $\hat{f}_1$ and $\hat{f}_2$ *equivalent* ($\hat{f}_1 \sim \hat{f}_2$) if there is are representatives $\mathcal{W}$ of $\mathcal{U}$ and $\mathcal{W}'$ of $\mathcal{U}'$ together with lifts of the identity $\varepsilon_i \in \mathrm{Orb}(\mathcal{W}, \mathcal{V}_i),\ i = 1, 2$ respectively $\varepsilon'_i \in \mathrm{Orb}(\mathcal{W}', \mathcal{V}'_i),\ i = 1, 2$ and a map $\hat{h} \in \mathrm{Orb}(\mathcal{W}, \mathcal{W}')$ such that the diagramm following diagramm commutes

$$\begin{tikzcd}
 & \mathcal{V}_1 \arrow[r, "\hat{f}_1"] & \mathcal{V}'_1 & \\
\mathcal{W} \arrow[ur, "\varepsilon_1"] \arrow[dr, "\varepsilon_2"'] \arrow[rrr, "\hat{h}"] & & & \mathcal{W}' \arrow[ul, "\varepsilon'_1"'] \arrow[dl, "\varepsilon'_2"] \\
 & \mathcal{V}_2 \arrow[r, "\hat{f}_2"] & \mathcal{V}'_2 &
\end{tikzcd}$$

Let $(Q,\mathcal{U})$ and $(Q',\mathcal{U}')$ be orbifolds. The notion of equivalence of charted maps induces an equivalence relation on the set of all charted orbifold maps whose domain atlas is contained in $\mathcal{U}$ and whose range family is contained in $\mathcal{U}'$. To prove this fact we need to clarify the relation of induced lifts and induced charted orbifold maps.

E.4.4 Lemma ([51, Lemma 5.13]) *Let $(Q,\mathcal{U})$ and $(Q',\mathcal{U}')$ be orbifolds and*

$$\hat{f} := (f, \{\, f_i \,\}_{i\in I}\,, [P,\nu]) \in \mathrm{Orb}(\mathcal{V},\mathcal{V}')$$

be a charted orbifold map, where $\mathcal{V}$ respectively $\mathcal{V}'$ is a representative of $\mathcal{U}$ respectively $\mathcal{U}'$. Suppose that there are orbifold charts $(V_\alpha, G_\alpha, \pi_\alpha) \in \mathcal{V}$, $\alpha = a, b$ and points $x_\alpha \in V_\alpha$ with $\pi_a(x_a) = \pi_b(x_b)$. Then there are arbitrarily small orbifold charts $(W, K, \chi) \in \mathcal{U}$, $(W', K', \chi') \in \mathcal{U}'$ and open embeddings $\lambda_\alpha \colon (W, K, \chi) \to (V_\alpha, G_\alpha, \pi_\alpha)$, $\mu_\alpha \colon (W', K', \chi') \to (V'_\alpha, G'_\alpha, \pi'_\alpha)$ with $x_\alpha \in \lambda_\alpha(W)$, $\alpha = a, b$ such that the induced lift g of f with respect to f_a, λ_a, μ_a coincides with the one induced by f_b, λ_b, μ_b. In other words, we obtain a commutative diagram

$$\begin{tikzcd}
 & V_a \arrow[r, "f_a"] & V'_a & \\
W \arrow[ur, "\lambda_a"] \arrow[dr, "\lambda_b"'] \arrow[rrr, "g"] & & & W' \arrow[ul, "\mu_a"'] \arrow[dl, "\mu_b"] \\
 & V_b \arrow[r, "f_b"] & V'_b &
\end{tikzcd}$$

E.4.5 Lemma ([51, Lemma 5.14]) *Let $(Q,\mathcal{U})$ and $(Q',\mathcal{U}')$ be orbifolds, $\mathcal{V}$ a representative of $\mathcal{U}$, and $\mathcal{V}'$ one of $\mathcal{U}'$. Further let $\hat{f} \in \mathrm{Orb}(\mathcal{V}, \mathcal{V}')$. Suppose that $\hat{h} \in \mathrm{Orb}(\mathcal{W}_1, \mathcal{W}'_1)$ and $\hat{g} \in \mathrm{Orb}(\mathcal{W}_2, \mathcal{W}'_2)$ are both induced by $\hat{f}$.*
There are representatives $\mathcal{W}$ of $\mathcal{U}$ and $\mathcal{W}'$ of $\mathcal{U}$ together with lifts of the identity $\varepsilon_i \in \mathrm{Orb}(\mathcal{W}, \mathcal{W}_i)$, $i = 1, 2$ and $\varepsilon'_i \in \mathrm{Orb}(\mathcal{W}', \mathcal{W}'_i)$, $i = 1, 2$, such that a charted orbifold map $\hat{k} \in \mathrm{Orb}(\mathcal{W}, \mathcal{W}')$ exists, making the following diagram commutative.

$$\begin{tikzcd}
 & \mathcal{W}_1 \arrow[r, "\hat{h}_1"] & \mathcal{W}'_1 & \\
\mathcal{W} \arrow[ur, "\varepsilon_1"] \arrow[dr, "\varepsilon_2"'] \arrow[rrr, "\hat{k}"] & & & \mathcal{W}' \arrow[ul, "\varepsilon'_1"'] \arrow[dl, "\varepsilon'_2"] \\
 & \mathcal{W}_2 \arrow[r, "\hat{g}_2"] & \mathcal{W}'_2 &
\end{tikzcd} \tag{E.4.2}$$

It follows from the last Lemma, that the relation $\sim$ introduced in definition E.4.3 is indeed an equivalence relation. For details we refer to the exposition in [51].

E.4.6 Definition Denote the equivalence class of a charted orbifold map $\hat{f}$ with respect to the equivalence relation $\sim$ introduced in Definition E.4.3 by $[\hat{f}]$. It will be clear from the context whether $\hat{f}$ is a charted orbifold map and $[\hat{f}]$ denotes its equivalence class, i.e. the orbifold morphism, or $\hat{f}$ is a representative of the charted orbifold map and $[\hat{f}]$ is equivalence class of representatives, which by abuse of notation is also abbreviated as $\hat{f}$.

E.5. The orbifold category

We have explained how to construct orbifolds and morphisms of orbifolds. Now we introduce the category of orbifolds, which is isomorphic to a full category of certain Lie groupoids (cf. [51] for details on this topic).

E.5.1 Definition The category **Orb** is defined as follows: The class of objects Ob **Orb** is given by the class of all paracompact Hausdorff orbifolds (as defined in Section 2.3.1). For two orbifolds $(Q,\mathcal{U})$ and $(Q',\mathcal{U}')$, the morphisms, i.e. *orbifold maps* from $(Q,\mathcal{U})$ to $(Q',\mathcal{U}')$ are the equivalence classes $[\hat{f}]$ of all charted orbifold maps $\hat{f} \in \mathrm{Orb}(\mathcal{V},\mathcal{V}')$ where $\mathcal{V}$ is a representative of $\mathcal{U}$ and $\mathcal{V}'$ is a representative of $\mathcal{U}'$, that is

$$\mathbf{Orb}((Q,\mathcal{U}),(Q',\mathcal{U}')) := \left\{ \left[\hat{f}\right] \middle| \hat{f} \in \mathrm{Orb}(\mathcal{V},\mathcal{V}'), \mathcal{V} \text{ representative of } \mathcal{U}, \mathcal{V}' \text{ representative of } \mathcal{U}' \right\}.$$

The composition in **Orb** is induced by the following construction: Let

$$[\hat{f}] \in \mathbf{Orb}((Q,\mathcal{U}),(Q',\mathcal{U}')) \quad \text{and} \quad [\hat{g}] \in \mathbf{Orb}((Q',\mathcal{U}'),(Q'',\mathcal{U}''))$$

be orbifold maps. Choose representatives $\hat{f} \in \mathrm{Orb}(\mathcal{V},\mathcal{V}')$ of $[\hat{f}]$ and $\hat{g} \in \mathrm{Orb}(\mathcal{W},\mathcal{W}')$ of $[\hat{g}]$. Then find representatives $\mathcal{K},\mathcal{K}',\mathcal{K}''$ of $\mathcal{U},\mathcal{U}',\mathcal{U}''$ respectively and lifts of the identity $\varepsilon \in \mathrm{Orb}(\mathcal{K},\mathcal{V})$, $\varepsilon_1' \in \mathrm{Orb}(\mathcal{K}',\mathcal{V}')$, $\varepsilon_2' \in \mathrm{Orb}(\mathcal{K}',\mathcal{W}')$, $\varepsilon'' \in \mathrm{Orb}(\mathcal{K}'',\mathcal{W}'')$ together with charted orbifold maps $\hat{h} \in \mathrm{Orb}(\mathcal{K},\mathcal{K}')$, $\hat{k} \in \mathrm{Orb}(\mathcal{K}',\mathcal{K}'')$ such that the diagramm

$$\begin{tikzcd}
& \mathcal{V} \arrow[r, "\hat{f}"] & \mathcal{V}' & & \mathcal{W}' \arrow[r, "\hat{g}"] & \mathcal{W}' & \\
\mathcal{K} \arrow[ur, "\varepsilon"] \arrow[rrr, "\hat{h}"] & & & \mathcal{K}' \arrow[ul, "\varepsilon_1'"'] \arrow[ur, "\varepsilon_2'"] \arrow[rrr, "\hat{k}"] & & & \mathcal{K}'' \arrow[ul, "\varepsilon''"']
\end{tikzcd}$$

commutes. Define the composition of $[\hat{g}]$ and $[\hat{f}]$ as

$$[\hat{g}] \circ [\hat{f}] := [\hat{k} \circ \hat{h}]$$

E.5.2 Proposition ([51, Lemma 5.17 and Proposition 5.18]) *It is always possible to compose two orbifold maps in* $\mathbf{Orb}((Q,\mathcal{U}),(Q',\mathcal{U}'))$ *and* $\mathbf{Orb}((Q',\mathcal{U}'),(Q'',\mathcal{U}''))$ *and the composition in* **Orb** *is well-defined.*

All equivalence classes which of lifts of the identity coincide for a given orbifold $(Q,\mathcal{U})$. Hence the “indentity morphism” introduced in Definition E.3.5 is the identity morphism of $(Q,\mathcal{U})$ in **Orb**.

E.5.3 Proposition ([51, Proposition 5.19]) *Let* $(Q,\mathcal{U})$ *be an orbifold and* ε *a lift of* $\mathrm{id}_{(Q,\mathcal{U})}$. *Then the equivalence class* $[\varepsilon]$ *of* ε *consists precicesly of all lifts of* $\mathrm{id}_{(Q,\mathcal{U})}$. *Hence the “identity morphism”* $\mathrm{id}_{(Q,\mathcal{U})}$ *is the equivalence class* $[\varepsilon]$

F. Orbifold geodesics: Supplementary Results

In this Section we supply proofs for some of the more technical assertions in Section 5.1

F.0.1 Lemma (Lemma 5.1.4) *Let* $[\hat{c}] \in \mathbf{Orb}(\mathcal{I}, (Q, \mathcal{U}))$ *be an orbifold path and* $[a, b] \subseteq \mathcal{I}$ *some compact subset. There exists a charted orbifold map* $\hat{g} := (c|_{]x,y[}, \{ g_k | 1 \leq k \leq N \}, (P_g, \nu_g))$ *with* $x < a < b < y$ *and* $N \in \mathbb{N}$, *such that:*

1. $[\hat{c}]|_{]x,y[} = [\hat{g}]$,
2. $\operatorname{dom} g_k =]l(k), r(k)[$ *for each* $1 \leq k \leq N$, *such that*
$$x = l(1) < l(2) < r(1) < l(3) < r(2) < \cdots < l(N) < r(N-1) < r(N) = y$$
3. $P_g = \left\{ \mathrm{id}_{]l(N),r(N)[} \right\} \cup \left\{ \mathrm{id}_{]l(k),r(k)[}, \iota_k^{k+1}, (\iota_k^{k+1})^{-1} \middle| 1 \leq k \leq N-1 \right\}$, *where* ι_k^{k+1} *is the canonical inclusion* $]l(k+1), r(k)[\hookrightarrow]l(k+1), r(k+1)[$

Proof of Lemma 5.1.4. Consider a representative $\hat{c_k} = (c, \{ c_i | i \in I \}, (P_c, \nu_c))$ of $[\hat{c}]$, whose domain atlas is contained in $\mathcal{A}_{\mathcal{I}}$. As $[a, b] \subseteq \mathcal{I}$ is compact, there is a finite subset $F \subseteq I$ such that $[a, b] \subseteq \bigcup_{i \in F} \operatorname{dom} c_i$ holds. Set $x := \inf \bigcup_{i \in F} \operatorname{dom} c_i$ and $y := \sup \bigcup_{i \in F} \operatorname{dom} c_i$ and consider $\hat{c}|_{]x,y[}$. By construction for $i \in F$ the set $\operatorname{dom} c_i$ is contained in $]x, y[$. Apply Lemma E.4.2 with respect to the family of pairs $\{ (\mathrm{id}_{\operatorname{dom} c_i}, \mathrm{id}_{\operatorname{cod} c_i}) | i \in F \}$ to obtain a representative $\hat{h}$ of $[\hat{c}]|_{]x,y[}$. The family of lifts of $\hat{h}$ is $\{ c_i \}_{i \in F}$. As F is finite, we choose and fix a partition of $]x, y[$ by real numbers $l(k)', r(k)', 1 \leq k \leq N \in \mathbb{N}$ which are ordered as in 2., such that $]l(k)', r(k)'[\subseteq \operatorname{dom} c_{i_k}$ holds for some $k_i \in F$. Note that each inclusion $\iota_k \colon]l(k)', r(k)'[\hookrightarrow \operatorname{dom} c_{i_k}$ is a change of orbifold charts. Apply Lemma E.4.2 with respect to the family of pairs $\left\{ (\iota_k, \mathrm{id}_{\operatorname{cod} c_{i_k}}) \middle| k \in \mathbb{N} \right\}$ to obtain a representative $\hat{g}' = (c_{]x,y[}, \{ g'_k | 1 \leq k \leq N \}, P_{g'}, \nu_{g'})$ induced by $\hat{h}$.
Choose $\iota_k^{k+1} \in P_{g'}$ with $\operatorname{dom} \iota_k^{k+1} \subseteq]l(k), r(k)[$ and $\operatorname{cod} \iota_k^{k+1} =]l(k+1), r(k+1)[$. Set $\iota_{k+1}^{k} := (\iota_k^{k+1})^{-1}$, $l(1) := x, r(N) := y$ and

$$r(k) := \sup \operatorname{dom} \iota_k^{k+1}, l(k+1) := \inf \operatorname{dom} \iota_k^{k+1} \text{ for each } 1 \leq k \leq N-1.$$

By construction $]l(k), r(k)[\subseteq]l(k)', r(k)'[$ holds for $1 \leq k \leq N$. The numbers $l(k), r(k)$ are ordered as in 2., since the $l(k)', r(k)'$ were ordered in this way. Furthermore $]x, y[= \bigcup_{1 \leq k \leq N}]l(k), r(k)[$ is satisfied. With this choice of ι_k^{k+1} the quasi-pseudogroup P_g as defined in 3. generates the change of charts for $\{]l(k), r(k)[| 1 \leq k \leq N \}$. Define

$$\nu_g(\lambda) := \begin{cases} \mathrm{id}_{\operatorname{cod} c_{i_k}} & \text{if } \lambda = \mathrm{id}_{]l(k),r(k)[}) \\ \nu_{g'}(\iota_k^{k+1}) & \text{if } \lambda = \iota_k^{k+1} \\ \nu_{g'}(\iota_k^{k+1})^{-1} & \text{if } \lambda = (\iota_k^{k+1})^{-1} \end{cases}$$

to obtain a well-defined map $\nu_g \colon P_g \to \mathbf{\Psi}(\mathcal{U})$.
Apply Lemma E.4.2 with respect to the pairs $\{ (]l(k), r(k)[\hookrightarrow]l(k)', r(k)'[, \mathrm{id}_{\operatorname{cod} c_{i_k}}) | 1 \leq k \leq N \}$ to obtain a representative $\hat{g} = (c|_{]x,y[}, \{ g_k | 1 \leq k \leq N \}, (P, \nu))$ induced by $\hat{g}'$. Reviewing the construction, $(P_g, \nu_g) \sim (P, \nu)$ holds, whence we may replace the pair (P, ν) with (P_g, ν_g). Observe that in each step, we applied Lemma E.4.2. Thus $[\hat{g}] = [\hat{c}]|_{]x,y[}$ holds. $\square$

F.0.2 Corollary *If $[\hat{c}] \in \mathbf{Orb}(\mathcal{I}, (Q,\mathcal{U}))$ is an orbifold geodesic and $[a,b] \subseteq \mathcal{I}$ compact, then the restriction $[\hat{g}] = [\hat{c}]|_{]x,y[}$ with $x < a < b < y$ constructed in Lemma 5.1.4 is an orbifold geodesic.*

Proof. As observed in the proof of this Lemma, each refinement has been constructed by an application of Lemma E.4.2. In particular the lifts g_k of $\hat{g}$ are obtained by restricting geodesics to open subsets of their domains. Hence each g_k is a geodesic, defined on the orbifold chart $(]l(k), r(k)[, \{ \mathrm{id}_{]l(k),r(k)[} \}, \pi_k) \in \mathcal{A}_{\mathcal{I}}$, where $\pi_k \colon]l(k), r(k)[\to \mathcal{I}$ is the inclusion of sets. We conclude that $[\hat{g}]$ is an orbifold geodesic. □

F.0.3 Lemma *Consider representatives $\hat{c} := (c, \{ c_k \}_{k\in A}, P, \nu)$, $\hat{c}' = (c', \{ c'_r \}_{r\in B}, P', \nu')$ of orbifold geodesics in $\mathbf{Orb}(\mathcal{I}, (Q,\mathcal{U}))$, whose domain atlases are contained in $\mathcal{A}_{\mathcal{I}}$. Assume that the lifts satisfy $\operatorname{cod} c_k = U_k$ for $(U_k, G_k, \psi_k) \in \mathcal{U}$ respectively $\operatorname{cod} c'_r = W_r$ for $(W_r, H_r, \varphi_r) \in \mathcal{U}$. The following conditions are equivalent*

(a) $[\hat{c}] = [\hat{c}']$,
(b) *for any $t \in \mathcal{I}$, such that $t \in \operatorname{dom} c_k \cap \operatorname{dom} c'_r$ holds for $k \in A, r \in B$, there is a change of charts $\lambda^{k,r}_t \colon U_k \supseteq \operatorname{dom} \lambda^{k,r}_t \to W_r$ with $T_t\lambda_t c_k(1) = T_t c_r(1)$ (i.e. the initial vectors coincide)*
(c) *for any $t \in \mathcal{I}$, there is a pair $(k,r) \in A \times B$ and a change of charts $\lambda_t \colon U_k \supseteq \operatorname{dom} \lambda_t \to W_r$ such that $t \in \operatorname{dom} c_k \cap \operatorname{dom} c'_r$ and $T_t\lambda_t c_k(1) = T_t c_r(1)$ hold*
(d) *There are representatives $\hat{g} = (c, \{ c_k \}_{k\in I}, P_g, \nu_g)$ of $[\hat{c}]$ respectively $\hat{g}' = (c, \{ c_k \}_{k\in I}, P_g, \nu'_g)$ of $[\hat{c}']$ whose domain atlases are contained in $\mathcal{A}_{\mathcal{I}}$.*
In particular a geodesic arc in Q is uniquely determined by the initial vector.

Proof. "(a) ⇒ (b)" is a reformulation of Lemma 5.1.3 for orbifold geodesics. "(b) ⇒ (c)" is trivial. To check "(c) ⇒ (d)" , we construct representatives induced by $\hat{c}$ and $\hat{c}'$: The chart domains of the domain atlases of $\hat{c}$ and $\hat{c}'$ are intervals $I_k := \operatorname{dom} c_k$, $k \in A$ respectively. $J_r := \operatorname{dom} c'_r$, $r \in B$. Pick some $t_0 \in \mathcal{I}$ together with a pair $(k,r) \in A\times B$ satisfying the hypothesis of (c). There is $\lambda_{t_0} \in \mathcal{C}h_{U_k,W_r}$ with $T_{t_0}\lambda_{t_0} c_k(1) = T_{t_0} c'_r(1)$. Shrinking $\operatorname{dom} \lambda_{t_0}$, we may assume that the set $t_0 \in \operatorname{dom} \lambda_{t_0}$ is G_k-stable. Thus it induces an orbifold chart $(\operatorname{dom} \lambda_{t_0}, G_{k,\operatorname{dom} \lambda_{t_0}}, \psi_k|_{\operatorname{dom} \lambda_{t_0}}) \in \mathcal{U}$. As c_k is a geodesic, we may choose $\varepsilon_{t_0} > 0$ with $c_k([t_0 - \varepsilon_{t_0}, t_0 + \varepsilon_{t_0}]) \subseteq \operatorname{dom} \lambda_{t_0}$ and $[t_0 - \varepsilon_0, t_0 + \varepsilon_0] \subseteq J_r$. The change of charts λ_{t_0} is a Riemannian isometry, since $(Q,\mathcal{U},\rho)$ is a Riemannian orbifold. In particular λ_{t_0} maps geodesics of the totally geodesic submanifold $\operatorname{dom} \lambda_{t_0} \subseteq U_k$ to geodesics of W_r. Thus $\lambda_{t_0} \circ c_k \colon]t_0 - \varepsilon_{t_0}, t_0 + \varepsilon_{t_0}[\to W_r$ is a geodesic. Uniqueness of geodesics in Riemannian manifolds implies that $\lambda_{t_0} \circ c_k|_{]t_0-\varepsilon_{t_0},t_0+\varepsilon_{t_0}[} = c'_r|_{]t_0-\varepsilon_{t_0},t_0+\varepsilon_{t_0}[}$ holds, as their derivatives coincide in t_0. For the trivial orbifold $\mathcal{I}$ the set $C_{t_0} :=]t_0 - \varepsilon_{t_0}, t_0 + \varepsilon_{t_0}[\subseteq I_k \cap J_r$ induces an orbifold chart via the inclusion of sets. Set $\alpha(t_0) := k$ and $\beta(t_0) := r$ and define change of orbifold charts $\mu_{t_0,\alpha} \colon C_{t_0} \to I_{\alpha(t_0)}$, $\mu_{t_0,\beta} \colon C_{t_0} \to J_{\beta(t_0)}$ and $\nu_{t_0,\alpha} \colon \operatorname{dom} \lambda_{t_0} \to U_{\alpha(t_0)}$ via the inclusion of sets. Furthermore set $\nu_{t_0,\beta} := \lambda_{t_0}$. Reviewing the construction, $c_{\alpha(t_0)}\mu_{t_0,\alpha} \subseteq \operatorname{Im} \nu_{t_0,\alpha}$ and $c'_{\beta(t_0)}\mu_{t_0,\beta} \subseteq \operatorname{Im} \lambda_{t_0} = \operatorname{Im} \nu_{t_0,\beta}$ hold. This implies

$$\nu^{-1}_{t_0,\alpha} c_{\alpha(t_0)}\mu_{t_0,\alpha} = \nu^{-1}_{t_0,\beta} c'_{\beta(t_0)}\mu_{t_0,\beta}. \tag{F.0.1}$$

With respect to the pair $(C_{t_0}, \{ \mathrm{id}_{C_{t_0}} \}, C_{t_0} \hookrightarrow \mathcal{I})$ and $(\operatorname{dom} \lambda_{t_0}, G_{k,\operatorname{dom} \lambda_{t_0}}, \psi_k|_{\operatorname{dom}_{t_0}})$ the lifts of $\hat{c}$ and $\hat{c}'$ coincide. The construction did not depend on t_0 and may be repeated for each $t \in \mathcal{I}$. In this way we obtain a (possibly infinite) subset $R \subseteq \mathcal{I}$, such that $\bigcup_{t\in R} C_t = \mathcal{I}$ and $C_t \neq C_s$ holds if $t \neq s$.

Since these sets cover $\mathcal{I}$, the construction yields an orbifold atlas $\mathcal{C} \subseteq \mathcal{A}_{\mathcal{I}}$ for $\mathcal{I}$. It may happen that the charts $(\operatorname{dom}\lambda_t, G_{\alpha(t),\operatorname{dom}\lambda_t}, \psi_{\alpha(t)}|_{\operatorname{dom}\lambda_t})$ and $(\operatorname{dom}\lambda_s, G_{\alpha(s),\operatorname{dom}\lambda_s}, \psi_{\alpha(s)}|_{\operatorname{dom}\lambda_s})$ coincide for $s \neq t$. To satisfy the requirement (R2) in Definition E.4.2, we redefine the charts: Set $\operatorname{dom}\lambda_s \times \{ s \}$ and redefine the group action, change of charts etc. in the obvious way. Without loss of generality we may thus assume $(\operatorname{dom}\lambda_t, G_{\alpha(t),\operatorname{dom}\lambda_t}, \psi_{\alpha(t)}|_{\operatorname{dom}\lambda_t}) \neq (\operatorname{dom}\lambda_s, G_{\alpha(s),\operatorname{dom}\lambda_s}, \psi_{\alpha(s)}|_{\operatorname{dom}\lambda_s})$ for $s \neq t$. Using Lemma E.4.2 the charted maps $\hat{c}$ resp. $\hat{c}'$ induce representatives $\hat{h}$ respectively $\hat{h}'$ with respect to $\mathcal{C}$ and the atlas $\mathcal{W} := \left\{ (\operatorname{dom}\lambda_t, G_{\alpha(t),\operatorname{dom}\lambda_t}, \psi_{\alpha(t)}|_{\operatorname{dom}\lambda_t})) \middle| t \in R \right\} \subseteq \mathcal{U}$. From (F.0.1) we deduce that the lifts of $\hat{h}$ and $\hat{h}'$ coincide. The set $\mathcal{I}$ is paracompact, second countable and a metric space with respect to the absolute value. Hence we may choose a refinement of the domain atlas of $\hat{h}$ as follows: There is a sequence of real numbers in $\mathcal{I}$

$$\cdots < l(-1) < r(-2) < l(0) < r(-1) < l(1) < r(0) < l(2) < r(1) < \cdots$$

such that $]l(n), r(n)[$ is contained in some chart of the domain atlas of $\hat{h}$ for each $n \in \mathbb{Z}$. Apply an argument as in the proof of Lemma 5.1.4 (cf. Lemma F.0.1 in Appendix F) to obtain an (at most countable) covering of $\mathcal{I}$ by intervals I_k indexed by $\mathbb{Z}$, such that the following is satisfied:

1. $I_k \cap I_j \neq \emptyset$ if and only if $j \in \{ k-1, k, k+1 \}$, $k, j \in \mathbb{Z}$,
2. $\hat{h}$ induces a representative $\hat{g} := (c, \{ g_k \}_{k\in\mathbb{Z}}, P_g, \nu_g)$ of $[\hat{c}]$ and $\hat{h}'$ induces a representative $\hat{g} := (c', \{ g'_k \}_{k\in\mathbb{Z}}, P'_g, \nu'_g)$ of $[\hat{c}']$, such that $P_g = P_{g'}$ and $P_g = \left\{ \mathrm{id}_{]l(k),r(k)[}, \iota_k^{k+1}, (\iota_k^{k+1})^{-1} \middle| k \in \mathbb{Z} \right\}$ hold, where $\iota_k^{k+1}, , (\iota_k^{k+1})^{-1}$ are defined as in Lemma 5.1.4.
3. As the lifts of $\hat{h}$ and $\hat{h}'$ coincide, for each $k \in \mathbb{Z}$ the lifts g_k, g'_k are given as restriction $g_k \colon]l(k), r(k)[\to V_k$, $(V_k, G_k, \psi_k) \in \mathcal{U}$ of a lift of $\hat{h}$.

Shrinking the sets $]l(n), r(n)[$, $n \in \mathbb{Z}$, we may assume that the images $g_k(]l(k+1), r(k)[)$ and $g_k(]l(k), r(k-1)[)$ are contained in stable subsets of $\operatorname{dom}\nu_{\hat{g}}(\iota_k^{k+1}) \cap \operatorname{dom}\nu_{\hat{g}'}(\iota_k^{k+1})$ respectively of $\operatorname{dom}\nu_{\hat{g}}((\iota_{k-1}^{k})^{-1}) \cap \operatorname{dom}\nu_{\hat{g}'}((\iota_{k-1}^{k})^{-1})$ for each $k \in \mathbb{Z}$. Restricting the change of charts to these stable subsets, by Defintion E.2.5 the pairs $(P_g, \nu_{\hat{g}})$ and $(P_g, \nu_{\hat{g}'})$ may be replaced by equivalent pairs, such that the maps $\nu_{\hat{g}}(\lambda), \nu_{\hat{g}'}(\lambda)$ are embeddings of orbifold charts with $\operatorname{dom}\nu_{\hat{g}}(\lambda) = \operatorname{dom}\nu_{\hat{g}'}(\lambda)$ for each $\lambda \in P_g$. Unfortunately $\nu_{\hat{g}}$ and $\nu_{\hat{g}'}$ need not coincide. However since the lifts coincide we obtain

$$\nu_{\hat{g}}(\iota_k^{k+1}) \circ g_k|_{]l(k+1),k[} = g_{k+1} \circ \iota_k^{k+1} = \nu_{\hat{g}'}(\iota_k^{k+1}) \circ g_k|_{]l(k+1),k[}$$

Hence both geodesic arcs coincide and it is uniquely determined by the intial vector. As $\nu_{\hat{g}}(\iota_k^{k+1})$ and $\nu_{\hat{g}'}(\iota_k^{k+1})$ are embeddings of orbifold charts with the same domain, for each $k \in \mathbb{Z}$ there is some $\gamma_{k+1} \in G_{k+1}$ with $\nu_{\hat{g}}(\iota_k^{k+1}) = \gamma_{k+1}.\nu_{\hat{g}'}(\iota_k^{k+1})$.
"(d) $\Rightarrow$ (a)" Consider representatives $\hat{g}$ of $[\hat{c}]$ and $\hat{g}'$ of $[\hat{c}']$ as constructed in Step "(b) $\Rightarrow$ (c)" . We claim that $[\hat{g}] = [\hat{g}']$ holds. To prove the claim, consider the case that the geodesic arc $\operatorname{Im} c$ contains non-singular points. Hence there are $k \in \mathbb{Z}$ and $z \in \mathcal{I}$ such that $c(z) = T\psi_k c_k(z)$ is non-singular. Each component of Σ_{G_k} is a totally geodesic submanifold of (V_k, ρ_k) by [40, II. Theorem 5.1]). Assume that there is an open, non-empty set U such that $\operatorname{Im} c_k \cap U$ is contained in a component of Σ_{G_k}. Then the image $\operatorname{Im} c_k$ is contained in this component (cf. [39, Proof of Theorem 1.10.15]). This contradicts the choice of $c_k(z)$, whence the non-singular points must be a dense subsete of $\operatorname{Im} c_k$ with respect to the subspace topology. Change of charts morphisms preserve non-singular points. Hence the same argument may be repeated to prove that the non-singular points must be dense in the image of each $c_k, k \in \mathbb{Z}$. In conclusion we have to consider two cases:

Case 1: The geodesic arc of $[\hat{c}]$ (or equivalently the arc of $[\hat{c}']$) contains a non-singular point. The preparatory considerations show that the non-singular points are dense in the image of each lift. Hence $\gamma_{k+1}.\nu_{\hat{g}}(\iota_k^{k+1}) = \nu_{\hat{g}'}(\iota_k^{k+1})$ implies $g_{k+1} = \mathrm{id}_{V_{k+1}}, \forall k \in \mathbb{Z}$ as $\operatorname{Im} c_{k+1}$ contains non-singular points. We deduce $\nu_{\hat{g}} = \nu_{\hat{g}'}$, whence $\hat{g} = \hat{g}'$ follows.

Case 2: The geodesic arc of $[\hat{c}]$ (or equivalently the arc of $[\hat{c}']$) is contained in the singular locus of Q. We construct a representative of $[\hat{c}]$ which coincides with $\hat{g}'$. Apply Lemma E.4.2 with suitable change of charts to $\hat{g}$ and $\hat{g}'$, such that $(V_k, G_k, \psi_k) \neq (V_j, G_j, \psi_j)$ holds if $k \neq j$. Observe that for each choice $\{\, \eta_k \in G_k \,\}_{k\in\mathbb{Z}}$ the pairs $\left\{\, (\mathrm{id}_{]l(k),r(k)[}, \eta_k) \,\right\}_{k\in\mathbb{Z}}$ induce another representative $\hat{h}$ of $[\hat{c}]$ by Lemma E.4.2. Recall from the construction of $\hat{h} = (c, \{\, \eta_k \circ c_k \,\}_{k\in\mathbb{Z}}, P_h, \nu_h)$ the following details: As $\eta_k \in G_k$ is defined on V_k, we may choose $P_h = P_{\hat{g}}$ and ν_h is uniquely determined by the identity

$$\nu_h(\iota_k^{k+1}) = \eta_{k+1}^{-1}\nu_{\hat{g}}(\iota_k^{k+1})\eta_k|_{\eta_k^{-1}(\operatorname{dom}\nu_{\hat{g}}(\iota_k^{k+1}))}. \tag{F.0.2}$$

We claim that it is possible to inductively choose the family η_k, such that $\eta_k c_k = c_k$ and $\nu_h = \nu_{\hat{g}'}$ hold. Begin with $k = 0$. Since $\operatorname{dom}\nu_{\hat{g}}(\iota_{-1}^0) = \operatorname{dom}\nu_{\hat{g}'}(\iota_{-1}^0)$ holds (and these maps are embeddings of orbifold charts by Step "(c) $\Rightarrow$ (d)"), by Proposition 2.2.2 (d) there is $\gamma_0 \in G_0$ with $\nu_{\hat{g}}(\iota_{-1}^0) = \gamma_0.\nu_{\hat{g}'}(\iota_{-1}^0)$. The isometry γ_0 fixes the geodesic c_0 pointwise on the set $\operatorname{Im} c_0 \cap \operatorname{cod}\nu_{\hat{g}'}(\iota_{-1}^0)$ as

$$\gamma_0 c_0|_{]l(0),r(-1)[} = \gamma_0\nu_{\hat{g}'}(\iota_{-1}^0)c_{-1}|_{]l(0),r(-1)[} = \nu_{\hat{g}}(\iota_{-1}^0)c_{-1}|_{]l(0),r(-1)[} = c_0|_{]l(0),r(-1)[} \tag{F.0.3}$$

holds. Thus Σ_{γ_0} contains $\operatorname{Im} c_0 \cap \operatorname{cod}\nu_{\hat{g}'}(\iota_{-1}^0)$ and each component of Σ_{γ_0} is a totally geodesic submanifold by [39, Theorem 1.10.15]. Hence $\gamma_0.c_0 = c_0$ follows (cf. the proof of [39, Theorem 1.10.15]). Set $\eta_0 := \gamma_0^{-1}$ and $\eta_{-1} := \mathrm{id}_{V_{-1}}$ to obtain $\eta_0.c_0 = c_0$ and $\eta_{-1}.c_{-1} = c_{-1}$. Furthermore (F.0.2) yields $\nu_h(\iota_{-1}^0) = \eta_0^{-1}\nu_{\hat{g}}(\iota_{-1}^0)\,\mathrm{id}_{V_{-1}} = \gamma_0^{-1}\nu_{\hat{g}}(\iota_{-1}^0) = \nu_{\hat{g}'}(\iota_{-1}^0)$. Proceed by induction on $k \geq 1$: Consider $k \geq 1$ such that for $0 \leq l < k$ elements $\eta_l \in G_l$ have been chosen with

$$\eta_l.c_l = c_l \text{ and } \nu_h(\iota_{l-1}^l) = \eta_l^{-1}.\nu_{\hat{g}}(\iota_{l-1}^l)\eta_{l-1}|_{\operatorname{dom}\nu_{\hat{g}}(\iota_{l-1}^l)} = \nu_{\hat{g}'}(\iota_{l-1}^l).$$

We have to choose η_k with $\eta_k.c_k = c_k$ and $\nu_h(\iota_{k-1}^k) = \eta_k^{-1}\nu_{\hat{g}}(\iota_{k-1}^k)\eta_{k-1}$. Argue as in the case $k = 0$: Since the embeddings of orbifold charts share the same domain, there is $\gamma_k \in G_k$ with $\gamma_k.\nu_{\hat{g}}(\iota_{k-1}^k) = \nu_{\hat{g}'}(\iota_{k-1}^k)$. A computation as (F.0.3) shows that γ_k fixes $\operatorname{Im} c_k$ pointwise. Since $\operatorname{dom}\nu_{\hat{g}}(\iota_{k-1}^k)$ is G_{k_1}-stable and η_{k-1} fixes $\operatorname{Im} c_{k-1}$ pointwise, $\eta_{k-1}(\operatorname{dom}\nu_{\hat{g}}(\iota_{k-1}^k)) = \operatorname{dom}\nu_{\hat{g}}(\iota_{k-1}^k)$ holds. Thus we consider the embedding of orbifold charts $\lambda := \nu_{\hat{g}}(\iota_{k-1}^k)\eta_{k-1}|_{\operatorname{dom}\nu_{\hat{g}}(\iota_{k-1}^k)}$. Since $\operatorname{dom}\lambda = \operatorname{dom}\gamma_k\nu_{\hat{g}'}(\iota_{k-1}^k)$ holds, Proposition 2.2.2 (d) yields unique $h_k \in G_k$ with $\lambda = h_k.\gamma_k.\nu_{\hat{g}'}(\iota_{k-1}^k)$. Define η_k via the formula $\eta_k := h_k \cdot \gamma_k \in G_k$. We compute the following identities:

$$\nu_h(\iota_{k-1}^k) = \eta_k^{-1}.\nu_{\hat{g}}(\iota_{k-1}^k)\eta_{k-1}|_{\operatorname{dom}\nu_{\hat{g}}(\iota_{k-1}^k)} = \eta_k^{-1}\lambda = \eta_k^{-1}.\eta_k.\nu_{\hat{g}'}(\iota_{k-1}^k) = \nu_{\hat{g}'}(\iota_{k-1}^k)$$

$$\begin{aligned}\eta_k.c_k|_{]l(k),r(k-1)[} &= \eta_k.\nu_{\hat{g}'}(\iota_{k-1}^k) \circ c_{k-1}|_{]l(k),r(k-1)[} = \nu_{\hat{g}}(\iota_{k-1}^k)\eta_{k-1}.c_{k-1}|_{]l(k),r(k-1)[}\\ &= \nu_{\hat{g}}(\iota_{k-1}^k) \circ c_{k-1}|_{]l(k),r(k-1)[} = c_k|_{]l(k),r(k-1)[}\end{aligned}$$

Thus the isometry η_k fixes the geodesic c_k pointwise on $\operatorname{Im} c_k \cap \operatorname{cod}\nu_{\hat{g}'}(\iota_{k-1}^k)$, whence η_k fixes all of $\operatorname{Im} c_k$ pointwise. We may thus inductively choose elements in $G_k, k \geq 1$, with the required

properties. Observe that by (R4) (c) and (d) of Definition E.2.3 $\nu_{\hat{g}}(\iota_k^{k-1})|_{\mathrm{Im}\,\nu_{\hat{g}}(\iota_{k-1}^k)} = \nu_{\hat{g}}(\iota_{k-1}^k)^{-1}$ holds. Instead of choosing η_k for $k < 0$ such that $\eta_{k+1}^{-1}\nu_{\hat{g}}(\iota_k^{k+1})\eta_k|_{\mathrm{dom}\,\nu_{\hat{g}}(\iota_k^{k+1})} = \nu_{\hat{g}'}(\iota_k^{k+1})$ holds, it suffices to choose η_k with $\eta_k^{-1}\nu_{\hat{g}}(\iota_{k+1}^k)\eta_{k+1}|_{\mathrm{dom}\,\nu_{\hat{g}}(\iota_{k+1}^k)} = \nu_{\hat{g}'}(\iota_{k+1}^k)$. If we require that η_k fixes c_k pointwise, then an argument as in the case $k \geq 1$ allows us to inductively choose η_k for $k < -1$ with the desired properties. Summing up, there is a family $\{\,\eta_k\,\}_{k\in\mathbb{Z}}$, such that $\hat{h} = \hat{g}'$ holds, where $\hat{h}$ was constructed via Lemma E.4.2 with respect to the pairs $\left\{\,(\mathrm{id}_{]l(k),r(k)[},\eta_k)\,\right\}_{k\in\mathbb{Z}}$. By Lemma E.4.2 $\hat{g} \sim \hat{h} = \hat{g}'$. Hence in both cases $[\hat{c}] = [\hat{g}] = [\hat{g}'] = [\hat{c}']$ follows from Definition E.4.3. □

F.0.4 Lemma (Lemma 5.1.9) *Let $[\hat{c}] \in \mathbf{Orb}\,(\mathcal{I},(Q,\mathcal{U}))$ and $[\hat{c}'] \in \mathbf{Orb}\,(\mathcal{I}',(Q,\mathcal{U}))$ be orbifold geodesics, such that for some $x_0 \in \mathcal{I}\cap\mathcal{I}'$ their initial vectors coincide. There is an unique orbifold geodesic $[\hat{c}\vee\hat{c}'] \in \mathbf{Orb}\,(\mathcal{I}\cup\mathcal{I}',(Q,\mathcal{U}))$, such that*

(a) *$[\hat{c}\vee\hat{c}']|_{\mathcal{I}'} = [\hat{c}']$ and $[\hat{c}\vee\hat{c}']|_{\mathcal{I}} = [\hat{c}]$ hold,*
(b) *let $K \subseteq \mathcal{I}$ be a compact set and $\hat{c} \in [\hat{c}]$. There is $\hat{g} \in [\hat{c}\vee\hat{c}']$ together with an open set $K \subseteq U \subseteq \mathcal{I}$, such that $\hat{g}_U$ and $\hat{c}|_U$ are equivalent as charted maps. Here $\hat{g}_U$ is the charted map, whose lifts are the lifts of $\hat{g}$ with domain contained in U and $\hat{c}|_U \in [\hat{c}]|_U$ is obtained by Lemma E.4.2 with respect to the pairs $(U\cap\mathrm{dom}\,c_k \hookrightarrow \mathrm{dom}\,c_k, \mathrm{id}_{\mathrm{cod}\,c_k})$, c_k is a lift of $\hat{c}$.*

Proof of Lemma 5.1.9. As a first step, we construct an orbifold geodesic on $\mathcal{I}\cup\mathcal{I}'$, with the same initial vector at x_0: If $\mathcal{I}\subseteq\mathcal{I}'$ holds, we set $[\hat{c}\vee\hat{c}'] := [\hat{c}]$. If $\mathcal{I}'\subseteq\mathcal{I}$ holds, set $[\hat{c}\vee\hat{c}'] := [\hat{c}]'$. For these cases, assertion (a) follows from Proposition 5.1.8 (b). Interchanging the roles of $[\hat{c}]$ and $[\hat{c}']$ if necessesary, it suffices to consider the case $\mathcal{I} =]a,b[$ and $\mathcal{I}' =]x,y[$ with $a < x < b < y$.
Fix $t_0 \in]x,b[$ with $t_0 > x_0$. We construct an orbifold geodesic by gluing several pieces: Choose representatives $\hat{c} = (c,\{\,c_k\,\}_{k\in A},[P_{\hat{c}},\nu_{\hat{c}}])$ of $[\hat{c}]$ and $\hat{c}' = (c',\{\,c'_r\,\}_{r\in B},[P_{\hat{c}'},\nu_{\hat{c}'}])$ of $[\hat{c}']$. Since the initial vectors of $[\hat{c}]$ and $[\hat{c}']$ at x_0 coincide, they coincide at each point in $\mathcal{I}\cap\mathcal{I} =]x,b[$ by Proposition 5.1.8. Combine Lemma F.0.3 (d) with Lemma E.4.2 to obtain $k_{t_0}\in A, r_{t_0}\in B$ with $t_0 \in \mathrm{dom}\,c_{k_{t_0}} = \mathrm{dom}\,c'_{r_{t_0}} \subseteq]x,b[$, such that $c'_{r_{t_0}} = c_{k_{t_0}}$ holds. Lemma F.0.3 (c) implies that

$$c\vee c'\colon\,]a,y[\to Q, t\mapsto \begin{cases} c(t) & t\in]a,b[\\ c'(t) & t\in]x,y[\end{cases}$$

is a continuous map, as the arcs of $[\hat{c}]|_{]x,b[}$ and $[\hat{c}']|_{]x,b[}$ coincide. Restricting the lifts Lemma E.4.2 allows us to obtain representatives $\hat{c}|_{]a,t_0[}$ induced by $\hat{c}$ and $\hat{c}'|_{]t_0,y[}$ induced by $\hat{c}'$:
By construction $\hat{c}|_{]a,t_0[} = (c|_{]a,t_0[},\{\,g_k\,\}_{k\in R},(P_{]a,t_0[},\nu_{]a,t_0[}))$ is obtained by restriction of all data to the open set $]a,t_0[$, i.e: There is a map $\alpha\colon R\to A$, such that the lifts satisfy $g_k = c_{\alpha(k)}|_{\mathrm{dom}\,c_{\alpha(k)}\cap]a,t_0[}$. Each element in $P_{]a,t_0[}$ is constructed as the restriction of an element in $P_{\hat{c}}$ to an open subset of its domain and $\nu_{]a,t_0[}(\mu|_{\mathrm{dom}\,\mu\cap]a,t_0})$ $:= \nu_{\hat{c}}(\mu)$. As $U_{t_0} := \mathrm{dom}\,c_{k_{t_0}}\cap]a,t_0[\neq\emptyset$ holds, this chart is contained in the domain atlas $\mathcal{W}_{]a,t_0[}$ of $\hat{c}|_{]a,t_0[}$. Let $i\colon U_{t_0}\to\mathrm{dom}\,c_{k_{t_0}}$ be the inclusion of sets. Define change of charts as follows: For $\lambda\in P_{]a,t_0[}$ and $W\in\mathcal{W}_{]a,t_0[}$ we define

$$\lambda_{t_0} := \begin{cases} \lambda\circ(i|^{\mathrm{Im}\,i\cap i(\mathrm{dom}\,\lambda)})^{-1} & \text{if } \lambda\in\mathcal{C}h_{U_{t_0},W} \\ i\circ\lambda & \text{if } \lambda\in\mathcal{C}h_{W,U_{t_0}} \\ i\circ\lambda\circ(i|^{\mathrm{Im}\,i\cap i(\mathrm{dom}\,\lambda)})^{-1} & \text{if } \lambda\in\mathcal{C}h_{U_{t_0},U_{t_0}} \end{cases}$$

Each of these change of charts is well defined and $\lambda \neq \mu$ implies $\lambda_{t_0} \neq \mu_{t_0}$. Thus we may define $\nu_{t_0}(\lambda_{t_0}) := \nu_{]a,t_0[}(\lambda)$. Furthermore set $\nu_{t_0}(\mathrm{id}_{\mathrm{dom}\, c_{k_{t_0}}}) := \mathrm{id}_{V_{k_{t_0}}}$, $\nu_{t_0}(i) := \mathrm{id}_{V_{k_{t_0}}}$ and $\nu_{t_0}(i^{-1}) := \mathrm{id}_{V_{k_{t_0}}}$. We obtain a set of change of charts

$$C_{t_0} := \left\{ \lambda_{t_0} \middle| \lambda \in \mathcal{C}h_{U_{t_0},W} \cup \mathcal{C}h_{W,U_{t_0}} \cup \mathcal{C}h_{U_{t_0},U_{t_0}},\ W \in \mathcal{W}_{]a,t_0[} \right\} \cup \left\{ \mathrm{id}_{\mathrm{dom}\, c_{k_{t_0}}}, i, i^{-1} \right\}$$

Since $P_{]a,t_0[}$ is a quasi-pseudogroup, the construction implies that $C := C_{t_0} \sqcup P_{]a,t_0[}$ is a quasi-pseudogroup which generates $\mathbf{\Psi}(\mathcal{W}_{]a,x[} \cup \left\{ (\mathrm{dom}\, c_{k^x_{t_0}}, \left\{ \mathrm{id}_{\mathrm{dom}\, c_{k^x_{t_0}}} \right\}, \mathrm{dom}\, c_{k^x_{t_0}} \hookrightarrow]a, \sup \mathrm{dom}\, c_{k^x_{t_0}}[) \right\})$. Our previous observations imply that the map

$$\nu_C \colon C \to \mathbf{\Psi}(\mathcal{U}), \lambda \mapsto \begin{cases} \nu_{t_0}(\lambda) & \text{if } \lambda \in C_{t_0} \\ \nu_{]a,t_0[}(\lambda) & \text{if } \lambda \in P_{]a,t_0[} \end{cases}$$

is well defined. Consider $\hat{c}_{a,t_0} := (c|_{]a,\sup \mathrm{dom}\, c_{k_{t_0}}}, \{ \mathrm{dom}\, g_k \}_{k \in R} \cup \{ c_{k_{t_0}} \}, C, \nu_C)$. The map $\nu_{]a,t_0[}$ satisfies property (R4) of Definition E.2.3. Together with the definition of λ_{t_0} and ν_C this implies that ν_C satisfies the property (R4). Hence $\hat{c}_{a,t_0}$ is a representative of an orbifold map, such that each lift is a geodesic defined on a chart in $\mathcal{A}_{]a,\sup \mathrm{dom}\, c_{k_{t_0}}[}$. In other words $[\hat{c}_{a,t_0}]$ is an orbifold geodesic, whose initial vector at any point in its domain coincides with the corresponding one of $[\hat{c}]$. Note that in the domain atlas of $\hat{c}_{a,t_0}$ only $(\mathrm{dom}\, c_{k^x_{t_0}}, \left\{ \mathrm{id}_{\mathrm{dom}\, c_{k^x_{t_0}}} \right\}, \mathrm{dom}\, c_{k^x_{t_0}} \hookrightarrow]a, \sup \mathrm{dom}\, c_{k^x_{t_0}}[)$ intersects $[t_0, b[$. We may thus interpret this chart as an "adhesive joint" .
Repeat the construction for $\hat{c}'$: We obtain $\hat{c}_{t_0,y} := (c'|_{]\inf \mathrm{dom}\, c_{r'_{t_0}}, y[}, \{ h_k \}_{k \in S} \cup \left\{ c'_{r_{t_0}} \right\}, D, \nu_D)$. Again only the chart with domain $\mathrm{dom}\, c'_{r_{t_0}} = \mathrm{dom}\, c_{k_{t_0}}$ in its domain atlas which intersects $]a, t_0]$.
We will glue the geodesics $\hat{c}_{a,t_0}, \hat{c}_{t_0,y}$ at their "adhesive joints" to obtain a geodesic on $]a, y[$: With the exception of $\mathrm{id}_{\mathrm{dom}\, c_{k_{t_0}}} = \mathrm{id}_{\mathrm{dom}\, c_{r_{t_0}}}$, the quasi-pseudogroups C and D contain only change of charts, whose domains are contained in $]a, t_0[$ (for C) respectively in $]t_0, y[$ (for D). In particular $C \cap D = \left\{ \mathrm{id}_{\mathrm{dom}\, c_{k_{t_0}}} \right\}$ holds, whence we obtain a disjoint union:

$$C \cup D = \left\{ \mathrm{id}_{\mathrm{dom}\, c_{k_{t_0}}} \right\} \sqcup C \setminus \left\{ \mathrm{id}_{\mathrm{dom}\, c_{k_{t_0}}} \right\} \sqcup D \setminus \left\{ \mathrm{id}_{\mathrm{dom}\, c_{k_{t_0}}} \right\}.$$

Consider $\lambda, \mu \in C \cup D$. If $\lambda \in C \setminus D$ and $\mu \in D$, such that the composition is defined on some open subset of their domains, then $\mu = \mathrm{id}_{\mathrm{dom}\, c_{k_{t_0}}} \in C$ follows. Vice versa an analogous condition holds for elements in $D \setminus C$. Thus any pair in $C \setminus D \times D \setminus C$ may not be composed on any open subset of their respective domains. As both sets C, D are quasi-pseudogroups, $P^\star := C \cup D$ is a quasi-pseudogroup which generates the change of charts of the atlas whose domains are given by $\{ \mathrm{dom}\, h_s | s \in S \} \cup \{ g_k | k \in R \} \cup \left\{ \mathrm{dom}\, c'_{r_{t_0}} \right\}$. Define

$$\nu^\star(\lambda) := \begin{cases} \nu_D(\lambda) & \text{if } \lambda \in D \\ \nu_C(\lambda) & \text{if } \lambda \in C \end{cases}.$$

As $\nu_C(\mathrm{id}_{c_{k_{t_0}}}) = \mathrm{id}_{V_{k_{t_0}}} = \mathrm{id}_{V_{r_{t_0}}} = \nu_D(\mathrm{id}_{c_{k_{t_0}}})$ holds, the map $\nu^\star$ is well defined. Since ν_C and ν_D satisfy condition (R4) the same holds for $\nu^\star$ with respect to the lifts $\{ h_s | s \in S \} \cup \{ c_{k_{t_0}} \} \cup$

$\{ g_k | k \in R \}$. Hence $\hat{c}^\star := (c \vee c', \{ h_s | s \in S \} \cup \{ c_{k_{t_0}} \} \cup \{ g_k | k \in R \}, P^\star, \nu^\star)$ is a representative of an orbifold geodesic on $]a, y[$.
Observe that the intial vector of $\hat{c}^\star$ at x_0 coincides by construction with the intial vector of $[\hat{c}]$ at x_0. As the initial vector of $[\hat{c}]$ coincides with the one of $[\hat{c}']$ in x_0, $[\hat{c} \vee \hat{c}'] := [\hat{c}^\star]$ satisfies the assertion (a) by Proposition 5.1.8. To prove assertion (b), construct a representative of $[\hat{c} \vee \hat{c}']$ by gluing:
There is some orbifold geodesic $[\hat{g}]$ on $]a, y[$ whose initial vector at any point of $]a, b[$ coincides with the one of $[\hat{c}]$. Let $K \subseteq]a, b[$ be a compact subset and $\hat{c} \in [\hat{c}]$ with $\hat{c} = (c, \{ c_k \}_{k \in I}, P_c, \nu_c)$. Enlarging K, we may assume $K = [a_1, a_2]$ for numbers $a < a_1 < a_2 < b$. Choose $\varepsilon > 0$, such that $a < a_1 - \varepsilon < a_2 + \varepsilon < b$ holds. For $i = 1, 2$ there are $k_i \in I$ such that $a_1 - \varepsilon \in \operatorname{dom} c_{k_1}$ and $a_2 + \varepsilon \in \operatorname{dom} c_{k_2}$. Choose $b_i \in \operatorname{dom} c_{k_i} \cap]a_1 - \varepsilon, a_2 + \varepsilon[$ and $b_1 < a_1 - \frac{\varepsilon}{2} < a_2 + \frac{\varepsilon}{2} < b_2$. Since $[\hat{g}]|_{]a_1 - \varepsilon, a_2 + \varepsilon[} = [\hat{c}]|_{]a_1 - \varepsilon, a_2 + \varepsilon[}$ holds, we apply Lemma F.0.3 to choose a representative $\hat{g}$ of $[\hat{g}]$ with the following properties: There are intervals $a_1 - \varepsilon \in J_1 \subseteq \operatorname{dom} c_{k_1} \cap]a, b_1[$ and $a_2 + \varepsilon \in J_2 \subseteq]b_2, y[$, such that $c_{k_i}|_{J_i}$ is a lift of $\hat{g}$ for $i = 1, 2$.
Define $m := \min \{ b_2, \sup \operatorname{dom} c_{k_1} \}$ and $M := \max \{ b_1, \inf \operatorname{dom} c_{k_2} \}$. Then the estimates $b_2 \geq m > b_1$ and $b_1 \leq M < b_2$ hold. As in the proof of (a) we construct representatives of orbifold geodesics $\hat{g}_{a,m}$ on $]a, m[$ respectively $\hat{g}_{M,y}$ on $]M, y[$ such that the following holds:

- $[\hat{g}_{a,m}] = [\hat{g}]|_{]a,m[}$, $[\hat{g}_{M,y}] = [\hat{g}]|_{]M,y[}$,
- For $i = 1$ the map $c_{k_1}|_{]\inf J_1, m[}$ is a local lift of $\hat{g}_{a,m}$ and it is the only lift whose domain intersect $]a_1 - \varepsilon, y[$.
- For $i = 2$ the map $c_{k_2}|_{]M, \sup J_2[}$ is a local lift of $\hat{g}_{M,y}$ and it is the only lift whose domain intersect $]a, a_2 + \varepsilon[$.

Furthermore let $\hat{c}|_{]b_1, b_2[}$ be the representative of $[\hat{c}]|_{]b_1, b_2[}$ which is obtained via Lemma E.4.2 with respect to the pairs $(\operatorname{dom} c_k \cap]b_1, b_2[\hookrightarrow \operatorname{dom} c_k, \mathrm{id}_{\operatorname{cod} c_k})_{k \in I}$. Arguing as in the proof of (a), we construct a representative $\hat{h}$ of $[\hat{c}]|_{]\inf J_1, \sup J_2[}$ from $\hat{c}|_{]b_1, b_2[}$ such that the only charts in the domain atlas of $\hat{h}$, which intersect $\mathcal{I} \cup \mathcal{I}' \backslash]b_1, b_2[$ are $\operatorname{dom} c_{k_1} \cap]\inf J_1, m[$ and $\operatorname{dom} c_{k_2} \cap]M, \sup J_2[$. The lifts on these charts are given by the restriction of the maps c_{k_i}, $i = 1, 2$. Gluing together the three parts $\hat{g}_{a,m}, \hat{h}, \hat{g}_{M,y}$, one obtains a representative $\hat{k}$ of an orbifold geodesic on $]a, y[$. The initial vectors of $\hat{k}$ coincide on $]b_1, b_2[$ with the ones of $[\hat{c}]$, whence $\hat{h}$ is a representative of $[\hat{c} \vee \hat{c}']$ by Lemma F.0.3.
Consider the family of charts $\{ I_k \}_{k \in S}$ in the domain atlas of $\hat{h}$, such that $I_k \subseteq]b_1, b_2[$ holds. By construction the family $\{ I_k \}$ covers $]b_1, b_2[$, since this holds for the domain atlas of $\hat{c}|_{]b_1, b_2[}$. Set $U :=]b_1, b_2[$. Then $K \subseteq U \subseteq]a, b[$ holds. Replacing the pair $(P_{\hat{k}}, \nu_{\hat{k}})$ with an equivalent pair, we may assume that $P_{\hat{k}}$ contains only change of chart morphisms. Then $P_{\hat{k}_U} := P_{\hat{k}} \cap \mathbf{\Psi}(\{ I_k \}_{k \in S})$ is a quasi-pseudogroup which generates $\mathbf{\Psi}(\{ I_k \}_{k \in S})$. Set $\nu_{\hat{k}_U} := \nu_{\hat{k}} | P_{\hat{k}_U}$ and denote by g_k the lift of $\hat{k}$ on I_k. Observe that by construction of the representative $\hat{k}$, each g_k is a lift of $\hat{c}|_{]b_1, b_2[}$. We obtain an orbifold geodesic $\hat{k}_U := (c|_{]b_1, b_2[}, \{ g_r \}_{r \in S}, (P_{\hat{k}_U}, \nu_{\hat{k}_U}))$ on $]b_1, b_2[$. By construction the lifts of $\hat{k}_U$ and of $\hat{c}|_{]b_1 b_2[}$ coincide and the pairs $(P_{\hat{c}|_{]b_1, b_2[}}, \nu_{\hat{c}|_{]b_1, b_2[}})$ and $(P_{\hat{k}_U}, \nu_{\hat{k}_U})$ are equivalent Hence $\hat{k}_U$ and $\hat{c}|_{]b_1 b_2[}$ are equivalent as charted orbifold maps.
If $P_{\hat{c}|_{]b_1, b_2[}}$ contains only change of orbifold chart maps, the above construction even yields $P_{\hat{c}|_{]b_1, b_2[}} = P_{\hat{k}_U}$ and $\nu_{\hat{k}_U} = \nu_{\hat{c}|_{]b_1, b_2[}}$. □

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List of Symbols

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